

Relativistic collision operators for modeling noninductive current drive by waves

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A weakly relativistic Fokker–Planck operator for electron–electron collision was first used by Karney and Fisch to calculate the efficiencies of current drive by waves with fast phase velocity [C. F. F. Karney and N. J. Fisch, *Phys. Fluids* **28**, 116 (1985)]. The present work extends Karney and Fisch’s work by expressing the weakly relativistic collision operator in potential form, and working out a general Legendre expansion of the potential functions. This general Legendre expansion reproduces the results in Karney and Fisch’s paper and is useful in implementing the weakly relativistic operator in Fokker–Planck codes. To justify the use of the weakly relativistic collision operator for current drive applications under ITER conditions, a comparison is made of current drive efficiencies predicted by this operator and a fully relativistic collision operator. Good agreement between efficiencies predicted by these two models is found. This suggests that the weakly relativistic collision operator is sufficiently precise for modeling the current drive schemes under ITER conditions. © 2011 American Institute of Physics. [doi:10.1063/1.3551739]

I. INTRODUCTION

Noninductive current drive by waves is the generation of electric current in fusion devices by injecting electromagnetic waves.^{1–4} An important quantity characterizing current drive (CD) is the current drive efficiency, which is defined as the ratio of the current density generated to the wave power absorbed per unit volume by the plasma. In electron-based current drive schemes, such as lower-hybrid current drive and electron cyclotron current drive, the collision models describing electron–electron and electron–ion collisions are crucial in determining the CD efficiencies. The collision term of species a off background species b is usually expressed as an operator, $C(f_a, f_b)$, where f_a and f_b are the distribution functions of test and background particles, respectively. There are several models for the collision operator $C(f_a, f_b)$. In the nonrelativistic case, the classical formulation for $C(f_a, f_b)$ was given by Rosenbluth *et al.*⁵ In the relativistic case, there are several relativistic Fokker–Planck operators in use. One of these is the weakly relativistic Fokker–Planck operator first used by Karney and Fisch to calculate lower-hybrid and electron cyclotron current drive efficiencies.⁶ A more complete collision model is the fully relativistic collision operator first developed by Beliaev *et al.*⁷ and then reformulated in differential form by Braams and Karney.^{8,9}

In this work, we extend Karney and Fisch’s work on weakly relativistic collision operators⁶ by expressing the weakly relativistic collision operator in potential form and deriving a general Legendre harmonics expansion of the potential functions. This general Legendre expansion can recover the results for the cases of low order Legendre harmonics reported in Karney and Fisch’s paper.⁶ Furthermore,

this general Legendre expansion can be used to calculate higher order Legendre harmonics cases, thus is useful in implementing the weakly relativistic operator in Fokker–Planck codes where the distribution functions are expanded to higher order Legendre harmonics.

The weakly relativistic collision operator can be obtained from the fully relativistic one by assuming either the test or the background species is weakly relativistic (i.e., $vv'/c^2 \ll 1$, where v and v' are respectively the velocities of test and background particles; c is the speed of light in vacuum). At the high electron temperature, $T_e \approx 25$ keV, expected for ITER,¹⁰ the weakly relativistic collision model is usually applicable. However, its accuracy in predicting CD efficiency should be established by comparing with a more complete model that fully considers relativistic effects in particle collisions. For the electron cyclotron current drive, the accuracy of weakly relativistic collision operator in predicting CD efficiency was considered in Ref. 11. Here, we consider the case of current drive by acceleration of electrons in the direction parallel to the equilibrium magnetic field. Three different cases of wave diffusion are considered, namely, Landau-damped, transit-time magnetic pumping, and Alfvén waves.^{6,12} CD efficiencies predicted by the weakly relativistic collision model are compared to those predicted by the fully relativistic one. In the cases considered, the CD efficiencies predicted by the weakly relativistic collision model agree well with those predicted by the fully relativistic one at the ITER electron temperature. The good agreement between the efficiencies determined by these two collision operators suggests that the weakly relativistic collision operator is sufficiently precise in predicting CD efficiencies under ITER conditions. Considering its simplicity over the fully relativistic one, the weakly relativistic collision operator is a con-

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venient and accurate collision model suitable to be used in rf heating and current drive applications.

The rest of this paper is organized as follows. Section II gives a brief review of relativistic collision operators. A general Legendre harmonics expansion of the potential form of the weakly relativistic collision operator is presented in Sec. III. Section IV gives a brief description of the adjoint method^{13–18} used in the calculation of CD efficiencies. Section V compares the CD efficiencies predicted by the weakly relativistic collision model with those predicted by the fully relativistic one. In addition, the effects of momentum conservation on the CD efficiencies are discussed. Conclusions are given in Sec. VI. Appendix A gives the details about the Legendre expansion. Details about the adjoint equation are given in Appendix B.

II. A BRIEF REVIEW OF RELATIVISTIC COLLISION OPERATORS

In the relativistic case, the collision term is usually expressed as the divergence of collision flux in momentum space,

$$C(f_a, f_b) = - \frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{S}^{ab}, \quad (1)$$

where \mathbf{u} is momentum per unit rest mass; \mathbf{S}^{ab} is the collision flux, which is given by the relativistic generalization of the Landau collision integral,^{6,7}

$$\mathbf{S}^{ab} = - \frac{c_{ab}}{m_a} \int \mathbf{U} \cdot \left(\frac{f_b(\mathbf{u}')}{m_a} \frac{\partial f_a(\mathbf{u})}{\partial \mathbf{u}} - \frac{f_a(\mathbf{u})}{m_b} \frac{\partial f_b(\mathbf{u}')}{\partial \mathbf{u}'} \right) d^3 \mathbf{u}', \quad (2)$$

where $c_{ab} = q_a^2 q_b^2 \ln \Lambda^{ab} / 8\pi\epsilon_0^2$, q_s is the charge of species s , $\ln \Lambda^{ab}$ is Coulomb logarithm, ϵ_0 is the vacuum dielectric constant, $f_s(\mathbf{u})$ is normalized so that $\int f_s(\mathbf{u}) d^3 \mathbf{u} = n_s$, with n_s being the number density, and \mathbf{U} is the collision kernel tensor. In the fully relativistic case, the collision kernel \mathbf{U} takes the form^{7–9}

$$\mathbf{U}(\mathbf{u}, \mathbf{u}') = \frac{r^2}{\gamma\gamma'w^3} [w^2 \mathbf{I} - \mathbf{u}\mathbf{u} - \mathbf{u}'\mathbf{u}' + r(\mathbf{u}\mathbf{u}' + \mathbf{u}'\mathbf{u})], \quad (3)$$

where \mathbf{I} is the unit tensor, γ is the Lorentz factor, $\gamma = \sqrt{1+u^2/c^2}$, $\gamma' = \sqrt{1+u'^2/c^2}$, $r = \gamma\gamma' - \mathbf{u} \cdot \mathbf{u}' / c^2$, $w = c\sqrt{r^2 - 1}$, $\mathbf{u} = \gamma\mathbf{v}$, $\mathbf{u}' = \gamma'\mathbf{v}'$. In the nonrelativistic limit, using $\gamma \rightarrow 1$, $\gamma' \rightarrow 1$, $r \rightarrow 1$, $w \rightarrow |\mathbf{v} - \mathbf{v}'|$, the collision kernel reduces to the simple Landau's form,¹⁹

$$\mathbf{U} = \frac{\mathbf{I}}{|\mathbf{v} - \mathbf{v}'|} - \frac{(\mathbf{v} - \mathbf{v}')(\mathbf{v} - \mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|^3}, \quad (4)$$

where \mathbf{v} and \mathbf{v}' are, respectively, the velocity of test and background particles. A careful examination of Landau's kernel reveals that it is only accurate to the zeroth order of v'/c (or v/c) of the fully relativistic kernel. We note in passing that a kernel accurate to the second order of v'/c (or v/c) had been developed by Pozzo *et al.*²⁰ to construct a more accurate weakly relativistic collision operator. The operator

in Eqs. (1) and (2) with the nonrelativistic limit kernel given by Eq. (4) is precisely the collision operator given by Landau.¹⁹ Indeed, an examination of Landau's derivation shows that the kinematics of the collisions are treated relativistically; the interaction, however, is calculated nonrelativistically assuming a Coulomb potential.⁶ In view of this physics behind the operator, it should be more appropriately referred to as a “semirelativistic” collision operator.^{6,8} The small parameter expansion used in getting the Landau's kernel from the fully relativistic kernel, however, usually leads to this operator being referred to as a “weakly relativistic” or “mildly relativistic” operator in the literature.²¹

The collision flux in Eq. (2) can also be written in the Fokker–Planck form,

$$\mathbf{S}^{ab} = - \mathbf{D}^{ab} \cdot \frac{\partial f_a(\mathbf{u})}{\partial \mathbf{u}} + \mathbf{F}^{ab} f_a(\mathbf{u}), \quad (5)$$

with the diffusion tensor \mathbf{D}^{ab} and friction vector \mathbf{F}^{ab} given respectively by

$$\mathbf{D}^{ab} = \frac{c_{ab}}{m_a^2} \int \mathbf{U} f_b(\mathbf{u}') d^3 \mathbf{u}', \quad (6)$$

$$\mathbf{F}^{ab} = - \frac{c_{ab}}{m_a m_b} \int \left(\frac{\partial}{\partial \mathbf{u}'} \cdot \mathbf{U} \right) f_b(\mathbf{u}') d^3 \mathbf{u}'. \quad (7)$$

For the nonrelativistic case, \mathbf{D}^{ab} and \mathbf{F}^{ab} can be expressed in terms of a pair of potential functions, and a Legendre harmonics expansion of these functions was developed by Rosenbluth *et al.*⁵ For the fully relativistic case, a similar work was done by Braams and Karney.^{8,9} However, in this case, six potential functions are involved in the formulation and the Legendre expansion is more complicated than the nonrelativistic case. For the weakly relativistic case, a similar work was done by Franz.²² In Sec. II, we present a work similar to Franz's for the weakly relativistic collision operator. The results obtained are in agreement with Franz's work; however, they are different from Franz's in the choice of potential functions.

III. POTENTIAL FORM AND LEGENDRE EXPANSION OF WEAKLY RELATIVISTIC COLLISION OPERATOR

In spite of the resemblance of the weakly relativistic collision operator with its nonrelativistic counterpart, the usual Rosenbluth potential approach⁵ does not apply. Here we construct two potential functions and express \mathbf{D}^{ab} and \mathbf{F}^{ab} in terms of these two functions. Furthermore, we derive a general Legendre harmonics decomposition of the potential functions.

Landau's collision kernel can also be written as $\mathbf{U} = \partial^2 |\mathbf{v} - \mathbf{v}'| / \partial \mathbf{v} \partial \mathbf{v}'$. Using this relation, one can easily put Eq. (6) into potential form. Define a potential function

$$h_b(\mathbf{v}) = -\frac{1}{8\pi} \int |\mathbf{v} - \mathbf{v}'| \gamma'^5 f_b(\gamma' \mathbf{v}') d^3 \mathbf{v}'. \quad (8)$$

Then, in terms of this potential, Eq. (6) can be written as

$$\mathbf{D}^{a/b}(\mathbf{u}) = -\frac{8\pi c_{ab}}{m_a^2} \frac{\partial^2 h_b(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}}. \quad (9)$$

Now, by expanding the background distribution f_b and the potential h_b in terms of Legendre polynomials $P_l(\cos \theta)$, under the assumption that f_b is axial symmetric about the magnetic field with θ being the included angle between velocity and the magnetic field,

$$f_b(u, \theta) = \sum_{l=0}^{\infty} f_b^l(u) P_l(\cos \theta), \quad (10)$$

$$h_b(v, \theta) = \sum_{l=0}^{\infty} h_b^l(v) P_l(\cos \theta), \quad (11)$$

we can get a relation between the expansion coefficients h_b^l and f_b^l ,

$$\begin{aligned} h_b^l(v) = & \frac{1}{2(4l^2 - 1)} \left[\int_0^v \frac{(v')^{l+2}}{v'^{l-1}} \left(1 - \frac{2l-1}{2l+3} \times \frac{(v')^2}{v^2} \right) \right. \\ & \times \gamma'^5 f_b^l(\gamma' v') dv' + \int_v^c \frac{v^l}{(v')^{l-3}} \left(1 - \frac{2l-1}{2l+3} \right. \\ & \left. \left. \times \frac{v^2}{(v')^2} \right) \gamma'^5 f_b^l(\gamma' v') dv' \right]. \quad (12) \end{aligned}$$

The above derivation is similar to the nonrelativistic case. In fact, compared with the nonrelativistic Rosenbluth potential,⁵ the only difference is the additional γ'^5 factor before $f_b(\gamma' \mathbf{v}')$ in Eq. (8) (this factor appears because

$d^3 \mathbf{u}' = \gamma'^5 d^3 \mathbf{v}'$). This difference does not influence the process of Legendre expansion. By this reasoning, we find that the potential in Eq. (8) is an analog of the original Rosenbluth potential.⁵ However, this analogy does not apply for deriving a potential form for the friction vector in Eq. (7). In this case, we need to calculate the divergence of the collision kernel in \mathbf{u}' space. After some algebra, we get

$$\frac{\partial}{\partial \mathbf{u}'} \cdot \mathbf{U} = -\frac{\partial}{\partial \mathbf{v}} \left[\frac{1}{|\mathbf{v} - \mathbf{v}'|} \left(\frac{1}{\gamma'} + \frac{1}{\gamma'^3} \right) + \frac{(\mathbf{v} \cdot \mathbf{v}' - v'^2)^2}{|\mathbf{v} - \mathbf{v}'|^3 \gamma'} \right]. \quad (13)$$

Define a potential function

$$\begin{aligned} g_b(\mathbf{v}) \equiv & -\frac{1}{4\pi} \int \left[\frac{1}{|\mathbf{v} - \mathbf{v}'|} \left(\frac{1}{\gamma'} + \frac{1}{\gamma'^3} \right) \right. \\ & \left. + \frac{(\mathbf{v} \cdot \mathbf{v}' - v'^2)^2}{|\mathbf{v} - \mathbf{v}'|^3 \gamma'} \right] \gamma'^5 f_b(\gamma' \mathbf{v}') d^3 \mathbf{v}'. \quad (14) \end{aligned}$$

Then, in terms of this potential, Eq. (7) can be written as

$$\mathbf{F}^{a/b}(\mathbf{u}) = -\frac{4\pi c_{ab}}{m_a m_b} \frac{\partial g_b(\mathbf{v})}{\partial \mathbf{v}}. \quad (15)$$

If the potential function g_b is expanded in terms of Legendre polynomials,

$$g_b(v, \theta) = \sum_{l=0}^{\infty} g_b^l(v) P_l(\cos \theta), \quad (16)$$

we can derive a relation between g_b^l and f_b^l . Here we write down the final expression for this relation, and give the details of this derivation in Appendix A. The final expression is

$$\begin{aligned} g_b^l(v) = & -\frac{1}{2l+1} \left\{ \int_0^v \left[\frac{1}{c^2} \frac{v'^{l+2}}{v'^{l-1}} \gamma'^2 \frac{l(l-1)}{2l-1} - \frac{1}{c^2} \frac{v'^{l+4}}{v'^{l+1}} \gamma'^2 \left(\frac{l^2+l-1}{2l+3} \right) + \frac{(v')^{l+2}}{v'^{l+1}} (\gamma'^2 + 1) \right] \gamma'^2 f_b^l(\gamma' v') dv' \right. \\ & \left. + \int_v^c \left[-\frac{1}{c^2} \frac{v^{l+2}}{v'^{l-1}} \gamma'^2 \frac{(l+1)(l+2)}{2l+3} + \frac{1}{c^2} \frac{v^l}{v'^{l-3}} \gamma'^2 \left(\frac{l^2+l-1}{2l-1} \right) + \frac{v^l}{(v')^{l-1}} (\gamma'^2 + 1) \right] \gamma'^2 f_b^l(\gamma' v') dv' \right\}. \quad (17) \end{aligned}$$

Equations (8), (9), (12), (14), (15), and (17) give the potential form and Legendre harmonics decomposition of the weakly relativistic collision term. Applying this general Legendre expansion for the cases of $l=0$ and $l=1$, one can recover the results in Karney and Fisch's paper [Eqs. (4) and Eq. (7) in Karney's paper⁶]. For the case of $l=0$, i.e., the background distribution is isotropic, $f_b(\mathbf{u}) = f_b(u)$, the diffusion and friction coefficients are given respectively by

$$\begin{aligned} D_{uu}^{a/b} = & -\frac{8\pi c_{ab}}{m_a^2} \frac{\partial^2 h_b^0(v)}{\partial v \partial v} \\ = & \frac{4\pi \Gamma^{a/b}}{3n_b} \left[\int_0^u u'^2 \frac{(v')^2}{v^3} f_b(u') du' \right. \\ & \left. + \int_u^\infty u'^2 \frac{1}{v'} f_b(u') du' \right], \quad (18) \end{aligned}$$

$$\begin{aligned}
D_{\theta\theta}^{a/b} &= -\frac{8\pi c_{ab}}{m_a^2} \frac{1}{v} \frac{\partial h_b^0(v)}{\partial v} \\
&= \frac{4\pi\Gamma^{a/b}}{3n_b} \left\{ \int_0^u u'^2 \frac{1}{2v^3} [3v^2 - (v')^2] f_b(u') du' \right. \\
&\quad \left. + \int_u^\infty u'^2 \frac{1}{v'} f_b(u') du' \right\}, \quad (19)
\end{aligned}$$

$$\begin{aligned}
F_u^{a/b} &= -\frac{4\pi c_{ab}}{m_a m_b} \frac{\partial g_b^0(v)}{\partial v} \\
&= -\frac{4\pi\Gamma^{a/b}}{3n_b} \frac{m_a}{m_b} \left[\int_0^u u' \left(3v' - \frac{v'^3}{c^2} \right) \frac{1}{v^2} f_b(u') du' \right. \\
&\quad \left. + \int_u^\infty u' \frac{2v}{c^2} f_b(u') du' \right], \quad (20)
\end{aligned}$$

where $\Gamma^{a/b} = n_b q_a^2 q_b^2 \ln \Lambda^{a/b} / 4\pi\epsilon_0^2 m_a^2 m_b^2$. Equations (18)–(20) agree respectively with Eqs. (4b), 4(d), and (4c) in Karney and Fisch's paper.⁶ Figure 1 is a plot of the weakly relativistic diffusion and friction coefficients for collision off the Maxwellian background distribution as a function of u . Also included in Fig. 1 are Fokker–Planck coefficients predicted by the fully relativistic collision model. These results show that Fokker–Planck coefficients determined by the two mod-

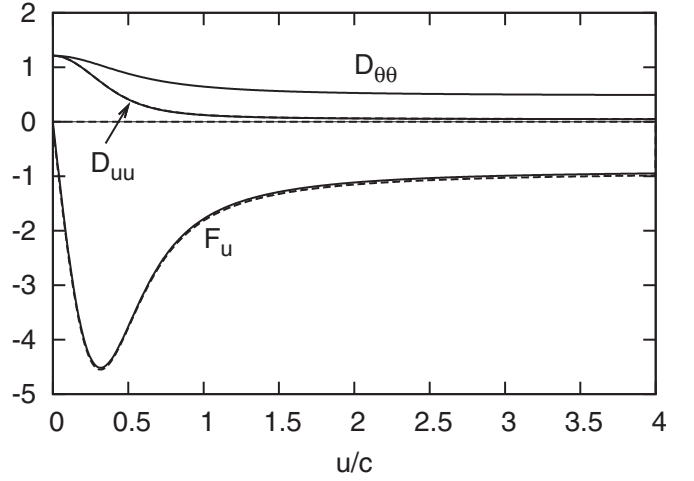


FIG. 1. Diffusion and friction coefficients for collision off relativistic Maxwellian distribution as a function of momentum per unit rest mass. Solid lines are coefficients predicted by the weakly relativistic collision model while dashed lines are those by the fully relativistic one. These are for electron–electron collision. Electron temperature is 25 keV. Diffusion coefficients, D_{uu} and $D_{\theta\theta}$, are normalized to $c^2\nu_c$; friction coefficient F_u is normalized to $c\nu_c$, where $\nu_c = \Gamma^{e/e}/c^3$.

els agree well with each other. For the case of $l=1$, here we give the explicit expression for the electron–electron collision term of the Maxwellian distribution off the first order Legendre harmonic,

$$\begin{aligned}
I(u) \equiv \frac{C[f_{em}, f_b^1(u) \cos \theta]}{f_{em} \cos \theta} &= \frac{4\pi\Gamma^{e/e}}{n_e} \left(\frac{1}{\gamma} f_b^1(u) + \frac{1}{5} \int_0^u u'^2 f_b^1(u') \frac{m_e}{T_e} \left\{ \frac{\gamma}{u^2} \frac{v'}{\gamma^3} \left[\frac{T_e}{m_e c^2} (4\gamma'^2 + 6) - \frac{1}{3} (4\gamma'^3 - 9\gamma') \right] \right. \right. \\
&\quad \left. \left. + \frac{\gamma^2 v'}{u^2 \gamma^3} \left[\frac{m_e v'^2}{T_e} \gamma^3 - \frac{1}{3} (4\gamma'^2 + 6) \right] \right\} du' + \frac{1}{5} \int_u^\infty u'^2 f_b^1(u') \frac{m_e}{T_e} \left\{ \frac{\gamma'}{u'^2} \frac{v}{\gamma^3} \left[\frac{T_e}{m_e c^2} (4\gamma^2 + 6) - \frac{1}{3} (4\gamma^3 - 9\gamma) \right] \right. \right. \\
&\quad \left. \left. + \frac{\gamma'^2 v}{u'^2 \gamma^3} \left[\frac{m_e v^2}{T_e} \gamma^3 - \frac{1}{3} (4\gamma^2 + 6) \right] \right\} du' \right). \quad (21)
\end{aligned}$$

Here f_{em} is the relativistic Maxwellian distribution of electrons with temperature T_e ,

$$f_{em}(u) = \frac{m_e n_e}{4\pi c T_e K_2(\Theta^{-1})} \exp\left(-\frac{\sqrt{1+u^2/c^2}}{\Theta}\right), \quad (22)$$

where m_e is the electron rest mass, $\Theta = T_e/m_e c^2$, and K_2 is the second order modified Bessel function of the second kind. Equation (21) agrees with Eq. (7) in Karney and Fisch's paper.⁶

IV. CURRENT DRIVE EFFICIENCY

In the linear theory of current drive, electron distribution function is assumed to be close to the Maxwellian distribution, so that the electron–electron collision term is approximated by a linearized collision operator, $C(f_e, f_e) \approx C(f_{e1}, f_{em}) + C(f_{em}, f_{e1})$, where f_{e1} is the perturbed distribu-

tion function, and f_{em} is the equilibrium Maxwellian distribution. In the widely used relativistic high-velocity limit collision operator,^{15,23} the linearized operator is further approximated by neglecting the $C(f_{em}, f_{e1})$ term, and expanding the first term in the high-velocity limit. Owing to its simplicity, this operator is often used in analytical works¹⁵ and is useful in predicting scaling laws of current drive efficiency.²³ However, the high-velocity limit operator does not conserve momentum for electron–electron collisions. This weakness makes it not accurate enough for the calculation of CD efficiencies, especially for the case of high electron temperatures.^{11,24} In this work, both terms in the linearized operator are retained [except for the case in Fig. 3 where the $C(f_{em}, f_{e1})$ term is discarded], and weakly and fully relativistic models are used to calculate these terms. Axial symmetry of the velocity distribution about the equilibrium magnetic field is assumed. To make this comparison study more

transparent, a uniform magnetic field equilibrium is used in the calculation of CD efficiencies. Thus, the trapped-particle effect found in toroidal geometry^{14,15} is not included in this formulation. In this case, the steady state of the perturbed distribution function of electrons is determined by the equation

$$C_e^l(f_{e1}) = \frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{S}_w + H, \quad (23)$$

where \mathbf{S}_w is the wave induced flux in momentum space, a quantity assumed to be known to us; f_{e1} is the perturbed electron distribution function with zero density and energy; $C_e^l(f_{e1})$ is the linearized collision operator taking into account of electron-electron and electron-ion collisions,

$$C_e^l(f_{e1}) = C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C^{ei}(f_{e1}). \quad (24)$$

The last term in Eq. (23) represents the slow (compared with the collision time) heating of the Maxwellian bulk, $H = (\varepsilon - \langle \varepsilon \rangle) T_e^{-2} f_{em} dT_e/dt$, where ε is the kinetic energy of electron, $\varepsilon = (\gamma - 1)m_e c^2$ and $\langle \varepsilon \rangle \equiv n_e^{-1} \int \varepsilon f_{em}(\mathbf{u}) d^3 \mathbf{u}$.

The electron-ion collision term $C^{ei}(f_{e1})$ in Eq. (24) is accurately modeled by the pitch angle scattering operator

$$C^{ei}(f_{e1}) = \Gamma^{ee} \frac{Z_i}{2v u^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f_{e1}}{\partial \theta} \right), \quad (25)$$

where Z_i is the effective ion charge and θ is the included angle between velocity and the magnetic field (this angle is usually called ‘‘pitch angle’’).

To evaluate the electron-electron collision term of the linearized collision operator defined in Eq. (24), two collision models will be used; one is the weakly relativistic collision operator first used by Karney and Fisch,⁶ and another is the fully relativistic collision model first developed by Beliaev *et al.*⁷ Both of these collision models ensure that momentum is conserved between particle collisions. Thus, the $C(f_{e1}, f_{em}) + C(f_{em}, f_{e1})$ term in Eq. (24) conserves momentum for electron-electron collisions.

To determine the driven current parallel to magnetic field from Eq. (23) using adjoint method,¹³⁻¹⁸ one needs to know the adjoint operator of C_e^l . The adjoint operator C_e^{l+} is related to C_e^l as $\int g C_e^{l+}(\psi) d^3 \mathbf{u} = \int \psi C_e^l(g) d^3 \mathbf{u}$, where $g(\mathbf{u})$ and $\psi(\mathbf{u})$ are two arbitrary functions. The linearized collision operators considered in this paper have the property that $\int g C_e^l(f_{em} \psi) d^3 \mathbf{u} = \int \psi C_e^l(f_{em} g) d^3 \mathbf{u}$. Using this property, one can construct the adjoint operator of C_e^l through

$$C_e^{l+}(\psi) = \frac{1}{f_{em}} C_e^l(\psi f_{em}). \quad (26)$$

We define the adjoint equation

$$C_e^{l+}(\chi) = e v_{\parallel}, \quad (27)$$

where v_{\parallel} is the velocity component parallel to the magnetic field, and we require that χf_{em} contains zero density and energy. If the solution to this equation is known, one can obtain the parallel current through the integration

$$j_{\parallel} = - \int \chi \frac{\partial}{\partial \mathbf{u}} \cdot \mathbf{S}_w d^3 \mathbf{u}. \quad (28)$$

The wave power absorbed per unit volume by the plasma is given by $P = - \int \varepsilon (\partial / \partial \mathbf{u} \cdot \mathbf{S}_w) d^3 \mathbf{u}$. The CD efficiency, defined as the ratio of j_{\parallel} to P , is then written as

$$\frac{j_{\parallel}}{P} = \frac{\int \mathbf{S}_w \cdot (\partial \chi / \partial \mathbf{u}) d^3 \mathbf{u}}{\int m_e \mathbf{v} \cdot \mathbf{S}_w d^3 \mathbf{u}}. \quad (29)$$

For current drive in uniform magnetic geometry, using the fact that Legendre polynomials are angular eigenfunctions of the collision operator C^l , it can be inferred that the solution to the adjoint equation, Eq. (27), consists of only the first Legendre harmonic, i.e., $\chi(\mathbf{u}) = \chi_1(u) \cos \theta$. Using this, the adjoint equation can be reduced to a one-dimensional (1-D) integro-differential equation about $\chi_1(u)$. The details of this 1-D integro-differential equation depend on the collision operator used and are provided in Appendix B.

V. COMPARISON OF CURRENT DRIVE EFFICIENCIES DETERMINED BY TWO RELATIVISTIC COLLISION MODELS

An important special case for CD efficiencies is when the wave induced flux is localized at a single point in momentum space, i.e., $\mathbf{S}_w \propto S_0 \delta(\mathbf{u} - \mathbf{u}_d) \hat{\mathbf{s}}$, where δ is the Dirac delta function. Then the CD efficiency, Eq. (29), reduces to

$$\frac{j_{\parallel}}{P} = \frac{\hat{\mathbf{s}} \cdot [\partial \chi / \partial \mathbf{u}]_{\mathbf{u}=\mathbf{u}_d}}{\hat{\mathbf{s}} \cdot m_e \mathbf{v}_d}, \quad (30)$$

where $\mathbf{v}_d = \mathbf{u}_d / \sqrt{1 + u_d^2/c^2}$ is the velocity of the deposition point. For the case of current drive by localized Landau-damped waves, the direction of wave induced flux \mathbf{S}_w is parallel to the equilibrium magnetic field, i.e., $\hat{\mathbf{s}} = \hat{\mathbf{u}}_{\parallel}$. We consider the case that the deposition point \mathbf{u}_d has only a parallel component. The CD efficiency, Eq. (30), then reduces to

$$\frac{j_{\parallel}}{P} = \frac{\chi_1'(u_d)}{m_e v_d}, \quad (31)$$

where $\chi_1' = d\chi_1/du$. The results of Eq. (31) are plotted as a function of the momentum of the deposition point in Fig. 2. The CD efficiency is normalized to $e/(m_e c v_c)$, where $v_c = \Gamma^{ee}/c^3$. Two collision models are considered, i.e., the weakly relativistic operator and the fully relativistic one. The CD efficiencies are calculated for four values of $\Theta = T_e/m_e c^2 = 0.02, 0.05, 0.1, 0.2$, corresponding to electron temperatures $T_e = 10, 26, 51, \text{ and } 102$ keV. The results in Fig. 2 indicate that the weakly relativistic model agrees with the fully relativistic model at low temperatures and low momentum. With the temperature increasing, the weakly relativistic model tends to overestimate the efficiency in the region of large u_d . For ITER condition, $T_e \approx 25$ keV, corresponding to the case of $\Theta = 0.05$ in Fig. 2, the result indicates that the two collision models give nearly identical drive efficiencies, with the largest relative error of about 5% taking place at $u_d/c = 5$. Also in Fig. 2 is the power deposition as a function of deposition point, and the trend indicates

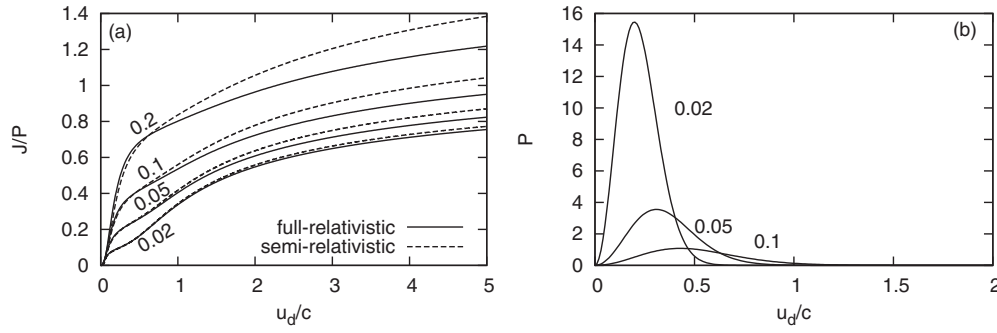


FIG. 2. (a) Current drive efficiencies (normalized to $e/m_e c v_e$) for localized excitation of Landau-damped waves as a function of the deposition point. The labels on the lines correspond to the temperature parameter $\Theta = T_e/m_e c^2$. The solid lines correspond to efficiencies predicted by the fully relativistic model, while the dashed lines correspond to those predicted by the weakly relativistic one. (b) Power deposition density $P \propto v_d^2 f_{em}(u_d)/\Theta$ as a function of the deposition point. In all cases, $Z_i = 1.67$.

that the power deposition at large momentum region is usually small.

In Fig. 3, as in Fig. 2, the CD efficiencies by localized excitation of Landau-damped waves are plotted as a function of the deposition point. Here the weakly relativistic collision operator is used to describe electron-electron collision. Efficiency calculated using the momentum-conserving linearized electron-electron collision term, $C(f_e, f_e) \approx C(f_{e1}, f_{em}) + C(f_{em}, f_{e1})$, and that calculated using the momentum nonconserving one, $C(f_e, f_e) \approx C(f_{e1}, f_{em})$, are compared for two cases of electron temperature, $T_e = 2$ and 25 keV. Results suggest that, in the case of $T_e = 2$ keV, efficiencies predicted by the momentum-conserving and non-conserving collision terms agree well with each other, while for $T_e = 25$ keV, the underestimation of the efficiencies by the momentum nonconserving model is appreciable. The physical explanation for the underestimation of the CD efficiencies by the momentum nonconserving model is that, in this model, part of the momentum (and hence current) is lost when the current-carrying electrons collide with the bulk electrons.²⁵

The second case considered is current drive by a narrow spectrum of lower-hybrid wave,^{26,27} for which the quasilinear diffusion tensor is given by^{25,28,29}

$$\mathbf{D}_w = \frac{\pi e^2}{2 m_e^2} |E_{\parallel}|^2 \delta(\omega - k_{\parallel} v_{\parallel}) \hat{\mathbf{u}}_{\parallel} \hat{\mathbf{u}}_{\parallel}, \quad (32)$$

where E_{\parallel} is the parallel electric field of the wave and ω and k_{\parallel} are, respectively, the wave frequency and parallel wave number. The wave induced flux in momentum space is related with \mathbf{D}_w by

$$\mathbf{S}_w = -\mathbf{D}_w \cdot \frac{\partial f_e}{\partial \mathbf{u}}. \quad (33)$$

In the linear theory of current drive, electron distribution is assumed to be weakly perturbed, so that we can take $f_e = f_{em}$ in Eq. (33) to give

$$\mathbf{S}_w = \frac{\pi e^2}{2 m_e^2} |E_{\parallel}|^2 \frac{1}{k_{\parallel}} \left(\frac{m_e}{T_e} \right) v_{\parallel} f_{em} \delta(v_{\parallel} - v_p) \hat{\mathbf{u}}_{\parallel}, \quad (34)$$

where $v_p = \omega/k_{\parallel}$ is the wave parallel phase velocity. The drive efficiency can be calculated from Eq. (29) to give

$$\frac{j_{\parallel}}{P} = \frac{1}{m_e v_p} \frac{\int_{u_{\min}}^{\infty} [G'(u)(v_p \gamma)^2/u + G(u)] u \gamma f_{em} du}{\int_{u_{\min}}^{\infty} \gamma f_{em} u du}, \quad (35)$$

where $u_{\min} = v_p / \sqrt{1 - v_p^2/c^2}$, $G(u) = \chi_1(u)/u$, and $G'(u) = dG/du$. The CD efficiencies [Eq. (35)] are plotted as a function of the phase velocity in Fig. 4. Results indicate that the weakly relativistic model agrees with the fully relativistic model at low temperature and low phase velocity. With the temperature increasing, the weakly relativistic model tends to overestimate the efficiency in the large phase velocity region. For ITER plasmas at $T_e \approx 25$ keV, corresponding to the case of $\Theta = 0.05$ in Fig. 4, the result suggests that the two collision models lead to nearly identical drive efficiencies, with the largest relative error of about 5% taking place at $v_p/c \approx 1$.

Similar calculations can be performed for transit-time magnetic pumping waves and Alfvén waves.⁶ These two waves both accelerate electrons in the direction parallel to

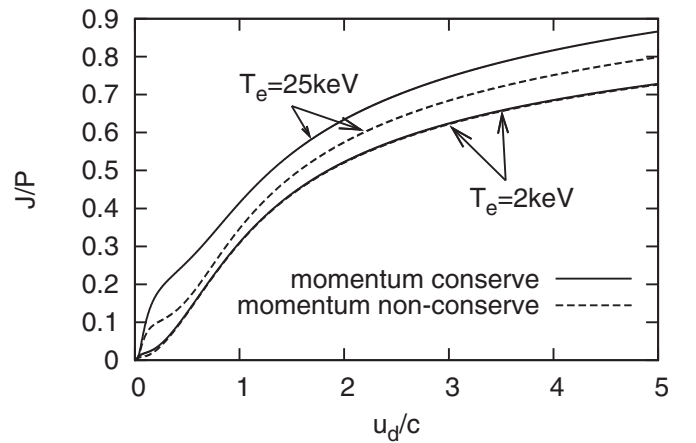


FIG. 3. Current drive efficiencies (normalized to $e/m_e c v_e$) predicted by the weakly relativistic collision model for localized excitation of Landau-damped waves as a function of the deposition point. The solid lines correspond to efficiencies calculated using the momentum-conserving collision term, $C(f_e, f_e) \approx C(f_{e1}, f_{em}) + C(f_{em}, f_{e1})$, while the dashed lines correspond to efficiencies calculated using the momentum nonconserving one, $C(f_e, f_e) \approx C(f_{e1}, f_{em})$. In all cases, $Z_i = 1.67$. Electron temperatures are indicated in the figure. Note that the two lines for the case of $T_e = 2$ keV are very close to each other.

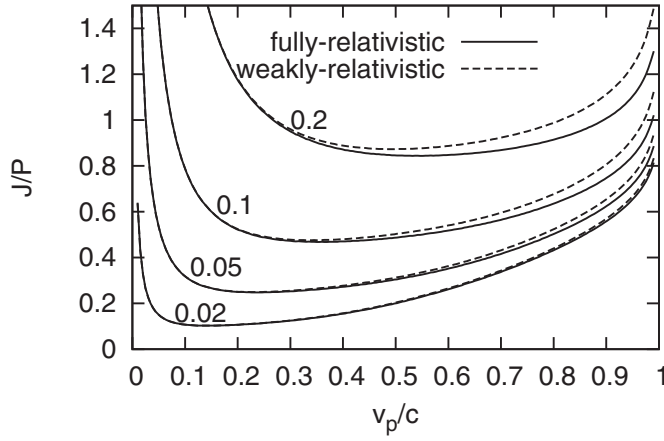


FIG. 4. Current drive efficiencies, normalized to $e/(m_e c v_e)$, for narrow spectrum of lower-hybrid waves as a function of phase velocity v_p . The labels on the lines correspond to the temperature parameter $\Theta = T_e/m_e c^2$. The solid lines correspond to the fully relativistic collision model while the dashed lines correspond to the weakly relativistic one. In all cases, $Z_i = 1.67$.

the equilibrium magnetic field. The quasilinear diffusion coefficients of these two waves differ from Landau-damped waves in their dependence on perpendicular velocity.^{6,12} For transit-time magnetic pumping waves, the quasilinear diffusion coefficients are

$$D_w \propto u_{\perp}^4 \delta(\omega - k_{\parallel} v_{\parallel}) \hat{u}_{\parallel} \hat{u}_{\parallel}. \quad (36)$$

For Alfvén waves,

$$D_w \propto (2T_e/m_e - u_{\perp}^2)^2 \delta(\omega - k_{\parallel} v_{\parallel}) \hat{u}_{\parallel} \hat{u}_{\parallel}. \quad (37)$$

The calculated CD efficiencies are plotted in Fig. 5 for transit-time magnetic pumping waves and in Fig. 6 for Alfvén waves. The results are similar to the case for the lower-hybrid wave. The CD efficiencies of transit-time magnetic pumping waves and Alfvén waves are almost the same,

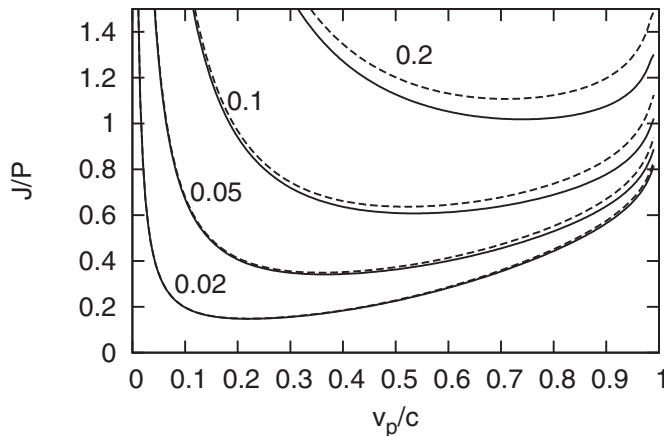


FIG. 5. Current drive efficiencies, normalized to $e/(m_e c v_e)$, for magnetic pumping waves as a function of phase velocity v_p . The labels on the lines correspond to the temperature parameter $\Theta = T_e/m_e c^2$. The solid lines correspond to the fully relativistic collision model while the dashed lines correspond to the weakly relativistic one. In all cases, $Z_i = 1.67$.

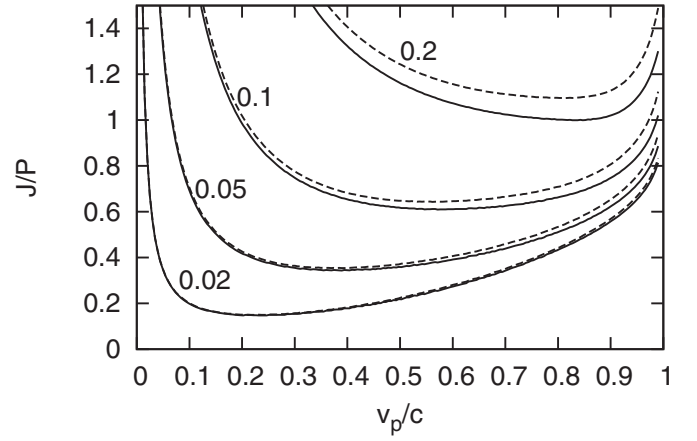


FIG. 6. Current drive efficiencies, normalized to $e/(m_e c v_e)$, for Alfvén waves as a function of phase velocity v_p . The labels on the lines correspond to the temperature parameter $\Theta = T_e/m_e c^2$. The solid lines correspond to the fully relativistic collision model while the dashed lines correspond to the weakly relativistic one. In all cases, $Z_i = 1.67$.

and both of them are larger than that of the lower-hybrid wave in the low phase velocity region. For the ITER electron temperature, $T_e \approx 25$ keV, corresponding to the case of $\Theta = 0.05$ in Figs. 5 and 6, the results indicate that the weakly and fully relativistic collision models predict nearly identical CD efficiencies for both waves.

VI. SUMMARY

We present here a general Legendre harmonics expansion for the potential form of a weakly relativistic collision operator. This general Legendre expansion is useful in implementing the weakly relativistic collision operator in Fokker-Planck codes. We also compare the current drive efficiencies predicted by the weakly relativistic collision model with those predicted by the fully relativistic one. In the cases considered, the drive efficiencies determined by the two collision models are in good agreement at the ITER electron temperature. This indicates that the weakly relativistic collision operator is sufficiently precise for modeling the current drive schemes under ITER conditions.

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APPENDIX A: DERIVATION OF EQUATION (17)

To get Eq. (17), we need to reduce the three-dimensional integration in the equation

$$g_b^j(v) \equiv -\frac{1}{P_j(\cos \theta)} \frac{1}{4\pi} \times \int d^3\mathbf{v}' \left[\frac{1}{|\mathbf{v}-\mathbf{v}'|} \left(\frac{1}{\gamma'} + \frac{1}{\gamma'^3} \right) + \frac{v'^2(v \cos \alpha - v')^2}{|\mathbf{v}-\mathbf{v}'|^3 \gamma'} \right] \gamma'^5 f_b^j(\gamma' v') P_j(\cos \theta') \quad (\text{A1})$$

to 1-D integration over v' . The task is to evaluate analyti-

$$\frac{(v \cos \alpha - v')^2}{|\mathbf{v}-\mathbf{v}'|^3} = (v \cos \alpha - v') \begin{cases} \sum_{l=0}^{\infty} l \frac{v'^{l-1}}{v^{l+1}} P_l(\cos \alpha), & \text{for } v' < v \\ -\sum_{l=0}^{\infty} (l+1) \frac{v^l}{v'^{l+2}} P_l(\cos \alpha), & \text{for } v' > v, \end{cases} \quad (\text{A3})$$

and

$$P_l(\cos \alpha) = \sum_{m=-l}^l \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta') P_l^m(\cos \theta) \times \exp(-im\phi'), \quad (\text{A4})$$

where P_l and P_l^m are the Legendre polynomial and associated Legendre function, respectively. Using these identities to expand the terms in Eq. (A1), after some algebra, one can obtain Eq. (17). Note that the final result of the right-hand side of Eq. (A1) is independent of the pitch angle θ .

APPENDIX B: ADJOINT EQUATION

The adjoint equation, Eq. (27), can be written as

$$\frac{C_e^d(\chi f_{em})}{f_{em} \cos \theta} = q_e v. \quad (\text{B1})$$

Knowing that the solution χ consists only of first Legendre harmonic, $\chi = \chi_1(u) \cos \theta$, the left-hand side of the above equation is written as

$$\frac{C_e^d(\chi f_{em})}{f_{em} \cos \theta} = [C(f_{em} \chi_1 \cos \theta, f_{em}) + C(f_{em}, f_{em} \chi_1 \cos \theta) + C^{ei}(f_{em} \chi_1 \cos \theta)] / f_{em} \cos \theta. \quad (\text{B2})$$

The third term on the right-hand side of Eq. (B2), electron-ion collision term, is given by the Lorentz limit

$$\frac{C^{ei}(f_{em} \chi_1 \cos \theta)}{f_{em} \cos \theta} = -\frac{Z_i \Gamma^{e/e}}{u^2 v} \chi_1. \quad (\text{B3})$$

The first term on the right-hand side of Eq. (B2) is given by

$$\begin{aligned} & \frac{C(f_{em} \chi_1 \cos \theta, f_{em})}{f_{em} \cos \theta} \\ &= \frac{1}{u^2} \frac{\partial}{\partial u} \left(u^2 D_{uu}^{e/e} \frac{\partial \chi_1}{\partial u} \right) + F_u^{e/e} \frac{\partial \chi_1}{\partial u} - \frac{2}{u^2} D_{\theta\theta}^{e/e} \chi_1. \end{aligned} \quad (\text{B4})$$

The expressions of the diffusion and friction coefficients,

cally the integration over ϕ' and θ' . Here (v', θ', ϕ') are the spherical coordinates of \mathbf{v}' ; α is the included angle between \mathbf{v} and \mathbf{v}' . We have the identities

$$\frac{1}{|\mathbf{v}-\mathbf{v}'|} = \begin{cases} \sum_{l=0}^{\infty} \frac{v'^l}{v^{l+1}} P_l(\cos \alpha), & \text{for } v' < v \\ \sum_{l=0}^{\infty} \frac{v^l}{v'^{l+1}} P_l(\cos \alpha), & \text{for } v' > v, \end{cases} \quad (\text{A2})$$

$D_{uu}^{e/e}$, $D_{\theta\theta}^{e/e}$, and $F_u^{e/e}$, depend on the collision operator used for electron-electron collision. For the fully relativistic case, $D_{uu}^{e/e}$, $D_{\theta\theta}^{e/e}$, and $F_u^{e/e}$ are given, respectively, by Eqs. (34a), (34b), and (34c) in Braams and Karney's paper.⁹ For the weakly relativistic case, these quantities are given by Eqs. (18)–(20) in this paper or equivalently by Eqs. (4) in Karney and Fisch's paper.⁶ The second term on the right-hand side of Eq. (B2) involves integration of $\chi_1(u)$. This term is given by Eq. (38) in Braams and Karney's paper⁹ for a fully relativistic case; for a weakly relativistic case, it is given by Eq. (21) in this paper (with f_b^1 replaced by $f_{em} \chi_1$) or equivalently by Eq. (7) in Karney and Fisch's paper.⁶

In this work, we adopt the numerical method proposed by Karney and Fisch⁶ to solve the adjoint equation. In this method, Eq. (B1) is casted as a 1-D diffusion equation by adding a time derivative $\partial \chi_1 / \partial t$ to the left-hand side of Eq. (B1) and we solve this diffusion equation until a steady state is reached. (The initial condition can be chosen arbitrarily.) The integration is carried out in the domain $0 < u < u_{\max}$, and the boundary conditions $\chi_1(0) = 0$ and $\chi_1'(u_{\max}) = 0$ are imposed. In the numerical implementation, the differential terms are treated implicitly, which makes large time step to be used. The integration term [the second term on the right-hand side of Eq. (B2)] is treated explicitly, which is recomputed after every time step.

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