Formulas for electron retrun current in neutral beam current drive for general tokamak equilibria and arbitrary collisionality regime

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The current driven by neutral beam injection (NBI) is the sum of the beam current carried by fast ions and the electron return current[1, 2, 3]. The electron return current (or called reverse/shielding current) is generated due to the momentum transfer from the fast ions to electrons. The return current in tokamak equilibrium is usually smaller than the one in uniform plasma due to the trapped particles effect in tokamak plasma[4, 5]. Previous theoretical calculation of the electron return current is usually limited to the case in which either the inverse aspect ratio or the electron collisionality is small. Lin-Liu and Hinton found that the ratio of the electron returen current to the fast ions current is closely related to one of the bootstrap current coefficients and use the exsiting formula for the coefficient which is valid in general tokamak equilibria but for low collisionality regime to epress the electron return current. In this report, by using the adjoint method, we extend the work in Ref.[5], which is valid for banana regime, to arbitrary collisionality regime. We show that the ratio of the electron return current to the fast ion current can still be expressed in terms of one of the bootstap current coefficients. We further make use of Sauter's bootstrap current coefficient formula[6], which is valid in general tokamka equilibra and arbitray collisionality regime, to give a convenient formula for calculating the electron return current.

In the presence of fast ions generated by NBI, the perturbed electron distribution satifies the following Fokker-Planck equation

$$v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla f_{e1} - C_l(f_{e1}) = C^{e/f}(f_{em}), \tag{1}$$

where ∇ is the gradient operator which is taken by holding the energy and magnetic moment constant, f_{em} and f_{e1} are electron equilibrium Maxwellian distribution and perturbed distribution function, respectively. $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, $\boldsymbol{B}$ is equilibrium magnetic field, $v_{\parallel}$ is electron velocity parallel to the magnetic field, $C_l(f_{e1}) = C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C^{e/i}(f_{e1})$ is the linearized collision term including electron-electron and electron-ion collision, $C^{e/f}(f_{em})$ is the collision term of electrons with fast ions, which is assumed to be known and acts as an inhomogeneous term in Eq. (1).

We want to determine the parallel (to the magnetic field) current density $j_{e\parallel}$ contributed by f_{e1}. It turns out that we can obtain $j_{e\parallel}$ via the following way. First solve the following adjoint equation

$$-v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla\chi_e - C_l(\chi_e) = q_e v_{\parallel} B f_{em} \tag{2}$$

to obtain the responce funtion χ_e, then $j_{e\parallel}$ can be expressed as

$$\left\langle j_{e\parallel} B\right\rangle = \left\langle \int d\Gamma \frac{\chi_e}{f_{em}} C^{e/f}(f_{em})\right\rangle. \tag{3}$$

where $\langle ...\rangle$ is the flux average. (The proof of Eq. (3) can be easily obtained by using the self-adjoint property of the operator $v_{\parallel}\hat{b}\cdot\nabla$ and C_l, i.e.,

$$\left\langle \int d\boldsymbol{v} g v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla h\right\rangle = -\left\langle \int d\boldsymbol{v} h v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla g\right\rangle, \tag{4}$$

and

$$\int d\boldsymbol{v} g C_l(f_{em} h) = \int d\boldsymbol{v} h C_l(f_{em} g), \tag{5}$$

where g and h are two arbitrary functions.)

In the usual situation of NBI, the fast ions beam velocity is much less than the electron thermal velocity, i.e. $u_f \ll v_{te}$. In this case, the collision term of electrons with fast ions can be approximated as

$$C^{e/f}(f_{em}) = \frac{m_e}{T_e}\nu_{ef} v_\| u_{f\|} f_{em}, \tag{6}$$

where $u_{f\|}$ are the parallel velocity of fast ions, $\nu_{ef} = Z_f^2 n_f \nu_{ei}/(Z_{\rm eff} n_e)$, here n_e and n_f are the number density of electrons and fast ions, respectively, Z_f is the charge number of fast ions, $Z_{\rm eff}$ is the effective charge number of plasma ions, $\nu_{ei} = \Gamma^{e/e} Z_{\rm eff}/v^3$ is the pitch angle scattering rate, $\Gamma^{e/e} = n_e e^4 \ln\Lambda^{e/e}/(4\pi\epsilon_0^2 m_e^2)$, $\ln\Lambda^{e/e}$ is the Coulomb logarithm, $-e$, m_e, and T_e are respectively the charge, mass, and temperature of electrons, ϵ_0 is the dielectric constant of free space. Using Eq. (6) in Eq. (3) gives

$$\langle j_{e\|} B\rangle = -\frac{Z_f}{Z_{\rm eff}}\frac{1}{I p_e}\left\langle j_{f\|} B \int d\Gamma \chi_e \nu_{ei}\frac{I v_\|}{\Omega_e}\right\rangle, \tag{7}$$

where $j_{f\|} = Z_f e n_f u_{f\|}$ is the fast ion current, $p_e = n_e T_e$, $\Omega_e = -Be/m_e$, $I = B_\varphi R$ is a flux surface function, B_φ is the toroidal magnetic field, R is the major radius. Ref. [6] points out that $\int d\Gamma \chi_e \nu_{ei} I v_\|/\Omega_e$ can be approximately considered to be a function of the flux surface. Using this, Eq. (7) is written as

$$\langle j_{e\|} B\rangle = -\langle j_{f\|} B\rangle \frac{Z_f}{Z_{\rm eff}}\frac{1}{I p_e}\left\langle \int d\Gamma \chi_e \nu_{ei}\frac{I v_\|}{\Omega_e}\right\rangle. \tag{8}$$

According to the neoclassical bootstap current theory of Sauter *et. al.*[6], (there is a minus sign error in Sauter's formula) we have

$$\frac{1}{I p_e}\left\langle \int d\Gamma \chi_e \nu_{ei}\frac{I v_\|}{\Omega_e}\right\rangle = 1 - \mathcal{L}_{31} \tag{9}$$

where $\mathcal{L}_{31}$ is the bootstrap current coefficient before the electron density gradient. Thus Eq. (8) is written as

$$\langle j_{e\|} B\rangle = -\langle j_{f\|} B\rangle \frac{Z_f}{Z_{\rm eff}}(1 - \mathcal{L}_{31}) \tag{10}$$

The formula of $\mathcal{L}_{31}$ given by Sauter *et al.*[6] is

$$\mathcal{L}_{31} = \left(1 + \frac{1.4}{Z_{\rm eff}+1}\right)X - \frac{1.9}{Z_{\rm eff}+1}X^2 + \frac{0.3}{Z_{\rm eff}+1}X^3 + \frac{0.2}{Z_{\rm eff}+1}X^4, \tag{11}$$

with

$$X = \frac{f_t}{1 + (1-0.1 f_t)\sqrt{\nu_{e\star}} + 0.5(1-f_t)\nu_{e\star}/Z_{\rm eff}}, \tag{12}$$

where $\nu_{e\star}$ is a measure of collisionality which is defined as $\nu_{e\star} = 0.012 n_{e20} Z_{\rm eff}\, qR/\varepsilon^{3/2} T_{e\,\rm keV}^2$, n_{e20} and $T_{e\,\rm keV}$ are electron number density in unit of $10^{20} m^{-3}$ and electron temperature T_e in unit of keV, respectively; q and ε are the safety factor and inverse aspect ratio of flux surface, respectively; f_t is the effective trapped fraction,

$$f_t = 1 - \frac{3}{4}\left\langle \frac{B^2}{B_{\max}^2}\right\rangle \int_0^1 \frac{\lambda d\lambda}{\left\langle \sqrt{1-\lambda B/B_{\max}}\right\rangle}. \tag{13}$$

We note that the formulas given by Eqs. (11)-(13) are valid for general tokamak equilibria and arbitrary collisionality regime. Thus, using these formulas in Eq. (10), we obtain a formula for the electron return current which is valid for general tokamak and arbitrary collisionality regime. The total current is the sum of the beam current carried by the fast ions and the electron return current, i.e. $j_\| = j_{f\|} + j_{e\|}$. Then we have

$$\langle j_\| B\rangle = \langle j_{f\|} B\rangle \left[1 - \frac{Z_f}{Z_{\rm eff}}(1 - \mathcal{L}_{31})\right], \tag{14}$$

and the ratio of the total current to the fast ion current

$$F \equiv \frac{\langle j_{\|} B \rangle}{\langle j_{f\|} B \rangle} = \left[1 - \frac{Z_f}{Z_{\rm eff}} (1 - \mathcal{L}_{31}) \right]. \tag{15}$$

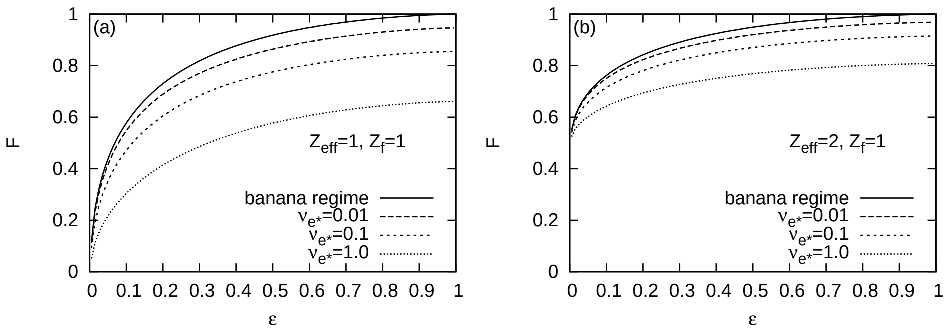

Figure 1. The ratio F of the total current to fast ion current [Eq. (15)] as a function of the inverse aspect ratio ε in concentric circluar flux surface equilibrium for $Z_f = 1$, $Z_{\rm eff} = 1$ (left figure), and $Z_{\rm eff} = 2$ (right figure). The different lines in the figure correspond to different values of the electron collision frequency, $\nu_{e\star} = 0.1$, 0.01, and 0.001. The results of the banana regime is obtained by using Eqs. (21)-26) in Ref.[5].

The difference between (a) and (b) of Fig. 1 is that in the former $Z_{\rm eff} = Z_f$, while in the latter $Z_{\rm eff} \neq Z_f$. When $Z_{\rm eff} = Z_f$ and the beam velocity is small, it is well known that the electron return current can cancel the fast ion current to make the net current zero in uniform plasmas. This result can be seen in Fig. 1(a) in the region $\varepsilon \to 0$.

In summary, we have showed that, for arbitrary aspect ratio and arbitrary collisionality regime, the ratio of the electron return current to the fast ion current in neutral beam current drive can be expressed in terms of electron density gradient coefficients of the bootstrap current, $\mathcal{L}_{31}$. Thus the existing formula for $\mathcal{L}_{31}$ valid for general tokamak equilibria and arbitrary collisionality regime provides a accurate formyla for calculating the electron return current. This formula for the electron return current can be easily applied to numerical codes modeling neutral beam current drive to improve the calculation capabilities of the codes.

One of the models of electron shielding current used in ONETWO[7] and TRANSP[8] trasport codes is a formula given by Hirshman[9, 10, 11, 12].

1 Hirshman's formula[11]

The ratio of the total current to the fast ion current is

$$F = 1 - \frac{Z_f}{Z_{\rm eff}} (1 - G). \tag{16}$$

Hirshman's fitting formula for G is

$$G = f_t \frac{\frac{3}{2} Z_{\rm eff} (K_{12} - \frac{5}{2} K_{11}) + (\sqrt{2} + \frac{13}{4} Z_{\rm eff}) K_{11}}{\left(\sqrt{2} + \frac{13}{4} Z_{\rm eff}\right) Z_{\rm eff} - \left(\frac{3}{2} Z_{\rm eff}\right)^2}, \tag{17}$$

where K_{11}, A_{11}, B_{11}, C_{11}, D_{11}, K_{12}, A_{12}, B_{12}, C_{12}, and D_{12} are given respectively by

$$K_{11}=\frac{0.53+Z_{\rm eff}}{(1+\sqrt{A_{11}\nu_{e\star}}+B_{11}\nu_{e\star})\Big(1+\sqrt{C_{11}\nu_{e\star}\varepsilon^{3/2}}+D_{11}\nu_{e\star}\varepsilon^{3/2}\Big)}, \tag{18}$$

$$A_{11}=3.44Z_{\rm eff}+\frac{0.52-0.42Z_{\rm eff}}{1+1.35Z_{\rm eff}}, \tag{19}$$

$$B_{11}=0.56+0.96Z_{\rm eff}, \tag{20}$$

$$C_{11}=0.25Z_{\rm eff}+\frac{0.14+0.55Z_{\rm eff}}{1+5Z_{\rm eff}}, \tag{21}$$

$$D_{11}=0.51Z_{\rm eff}+\frac{0.7+0.78Z_{\rm eff}}{1+Z_{\rm eff}}, \tag{22}$$

$$K_{12}=\frac{0.71+Z_{\rm eff}}{(1+\sqrt{A_{12}\nu_{e\star}}+B_{12}\nu_{e\star})(1+\sqrt{C_{12}\nu_{e\star}\varepsilon^{3/2}})+D_{12}\nu_{e\star}\varepsilon^{3/2}},$$

$$A_{12}=0.31Z_{\rm eff}+\frac{0.1+0.084Z_{\rm eff}}{1+1.3Z_{\rm eff}}, \tag{23}$$

$$B_{12}=0.26+0.35Z_{\rm eff}, \tag{24}$$

$$C_{12}=0.081Z_{\rm eff}+\frac{0.072+0.15Z_{\rm eff}}{1+3Z_{\rm eff}}, \tag{25}$$

$$D_{12}=0.29Z_{\rm eff}+\frac{0.42+0.62Z_{\rm eff}}{1+1.42Z_{\rm eff}}. \tag{26}$$

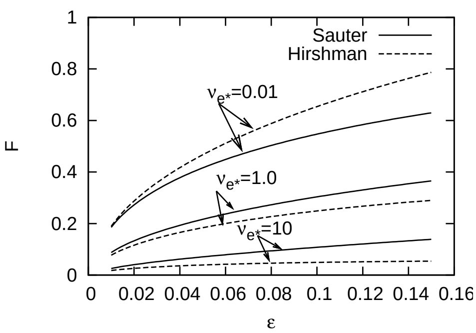

Figure 2. Comparison of the ratio F calculated by Sauter's formula and the one by Hirshman's. Hirshman's formula is valid only for $0.01 \leqslant \varepsilon \leqslant 0.15$, so the comparison limited in this range.

According to Wesson's book[13], electron-electron collision frequency ν_{ee} is defined as

$$\nu_{ee}\equiv\frac{\sqrt{2}}{12\pi^{3/2}}\frac{n_e e^4}{\varepsilon_0^2\sqrt{m_e}T_e^{3/2}}\ln\Lambda_e. \tag{27}$$

(The definition of ν_{ee} is different from Karney's, while it agrees with Eq. (56) in Kraus' notes, where ν_{ee} is written as

$$\nu_{ee}=\frac{16\sqrt{\pi}}{3}\frac{n_e e^4\ln\Lambda_e}{m_e^2 v_{te}^3}, \tag{28}$$

which is in Gaussian units. By using the trasforming rule, we replace ε_0 in Eq. (27) by $1/4\pi$, which gives Eq. (28). Also we note that Kraus's formula is identical with the ν_{ee} defined in Hirshman's paper[11] where $\nu_{ee}=1/\tau_{ee}$ and τ_{ee} is defined after Eq. (9) in Hirshman's paper)

The dimensionless collision parameter $\nu_{e\star}$ is defined as

$$\nu_{e\star} \equiv \frac{\nu_{ee}}{\varepsilon^{3/2} v_{te}/(qR)}, \tag{29}$$

where $v_{te} = \sqrt{T_e/m_e}$, q and ε are respectively the safety factor and inverse aspect ratio of a flux surface. Eq. () can be rewritten as

$$\begin{aligned}\nu_{e\star} &= \frac{\sqrt{2}}{12\pi^{3/2}} \frac{e^4}{\varepsilon_0^2} \frac{n_e q R}{T_e^2 \varepsilon^{3/2}} \ln\Lambda_e \\ &= \left(\frac{\sqrt{2}}{12\pi^{3/2}} \frac{e^2}{\varepsilon_0^2}\right) \frac{n_e q R}{\overline{T}_e^2 \varepsilon^{3/2}} \ln\Lambda_e,\end{aligned} \tag{30}$$

where $\overline{T}_e$ is electron temperatur in unit of eV, all other dimensional quantities are evaluated in SI units (i.e., normalized to SI unit). In SI unit we have

$$\frac{\sqrt{2}}{12\pi^{3/2}} \frac{e^2}{\varepsilon_0^2} = 6.92 \times 10^{-18}, \tag{31}$$

which agrees with constants appearing in Sauter's formula.

2 Beam driven current in general tokamak equilibria

This section is a review of Ref. [5]. We start with the basic equation

$$v_\parallel \hat{\boldsymbol{b}} \cdot \nabla f_1 - C^l(f_1) = -\boldsymbol{v}_d \cdot \nabla f_{em} + C(f_{em}, f_f), \tag{32}$$

where f_1 is the perturbed electron distribution function, $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, $\boldsymbol{B}$ is equilibrium magnetic field, $\boldsymbol{v}_d$ is the drift velocity of guiding-centers perpendicular to the magnetic field, $C^l(f_1) = C(f_1, f_{em}) + C(f_{em}, f_1) + C^{e/i}(f_1)$ is the linearized electron collision term including electron-electron and electron-ion collision, f_f is the fast ion distribution and $C(f_{em}, f_f)$ is the electron collisions with fast ions, which is known and acts as an inhomogeneous term of Eq. (32). We can eleminate the first term on the right-hand side of Eq. (32) by writing f_1 in the form

$$f_1 = -\frac{I v_\parallel}{\Omega_e} \frac{\partial f_{em}}{\partial \psi} + g, \tag{33}$$

where ψ is the label of the flux surface (here it is chosen to be poloidal flux), $I = B_\varphi R$, which is a function of only ψ, $\Omega = B q_e / m_e c$ (Ω_e includes the sign of the charge of electron). Using Eq. (33) in Eq. (32) gives

$$v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \left(-\frac{I v_\parallel}{\Omega_e} \frac{\partial f_{em}}{\partial \psi}\right) + v_\parallel \hat{\boldsymbol{b}} \cdot \nabla g - C^l(g) + C^l\left(\frac{I v_\parallel}{\Omega_e} \frac{\partial f_{em}}{\partial \psi}\right) = -\boldsymbol{v}_d \cdot \nabla f_M + C(f_{em}, f_f) \tag{34}$$

Noting that $I \partial f_M / \partial \psi$ is constant along a magnetic field line, the above equation becomes

$$-I \frac{\partial f_{em}}{\partial \psi} v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \left(\frac{v_\parallel}{\Omega_e}\right) + v_\parallel \hat{\boldsymbol{b}} \cdot \nabla g - C^l(g) = -\boldsymbol{v}_d \cdot \nabla f_{em} + C(f_{em}, f_f) - C^l\left(\frac{I v_\parallel}{\Omega_e} \frac{\partial f_{em}}{\partial \psi}\right) \tag{35}$$

Using

$$\boldsymbol{v}_d \cdot \nabla \psi = I v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \left(\frac{v_\parallel}{\Omega_e}\right) \tag{36}$$

(This is Eq. (7) in Lin-Liu's paper[5], refer to Sec. 9 for the proof of this identiy (however there is a minus difference betwwen mine and Lin-Liu's. I do not know why), then we obtain

$$\begin{aligned}\boldsymbol{v}_d \cdot \nabla f_{em} &= \frac{\partial f_{em}}{\partial \psi} \boldsymbol{v}_d \cdot \nabla \psi \\ &= \frac{\partial f_{em}}{\partial \psi} I v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \left(\frac{v_\parallel}{\Omega_e}\right)\end{aligned} \tag{37}$$

Using this in Eq. (35) gives

$$v_{\|}\hat{\boldsymbol{b}}\cdot\nabla g - C^l(g) = C(f_{em}, f_f) - C^l\left(\frac{I v_{\|}}{\Omega_e}\frac{\partial f_{em}}{\partial\psi}\right) \tag{38}$$

Next we give the specific form of the two collision terms on the right-hand side of Eq. (38). We note that (assuming temperature is uniform)

$$\frac{\partial f_{em}}{\partial\psi} = \frac{\partial n_e}{\partial\psi}\frac{1}{n_e}f_{em}. \tag{39}$$

Then the second inhomogeneous term of Eq. (38) is written as

$$\begin{aligned} C^l\left(\frac{I v_{\|}}{\Omega_e}\frac{\partial f_{em}}{\partial\psi}\right) &= \frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}C^l(v_{\|} f_{em}) \\ &= \frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}\left[C(v_{\|} f_{em}, f_{em}) + C(f_{em}, v_{\|} f_{em}) + C^{e/i}(v_{\|} f_{em})\right] \\ &= \frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}\left[C^{e/i}(v_{\|} f_{em})\right], \end{aligned} \tag{40}$$

where the last equality is due to $C(v_{\|} f_{em}, f_{em}) + C(f_{em}, v_{\|} f_{em}) = 0$. We approximate the electron-ion collision by the pitch-angle scatering operator, $C^{e/i}(h) \approx \nu_{ei}L(h)$, where $\nu_{ei}(v) = \Gamma^{e/e}Z_{\rm eff}/v^3$ is the scattering rate, L is the Legendre operator

$$L(h) \equiv \frac{1}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}h\right), \tag{41}$$

where θ is the pitch-angle, which is the included angle between the velocity and the magnetic field. Then Eq. (40) is written as

$$\begin{aligned} C^l\left(\frac{I v_{\|}}{\Omega_e}\frac{\partial f_{em}}{\partial\psi}\right) &= f_{em}v\frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}\nu_{ei}L(\cos\theta) \\ &= -f_{em}v\frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}\nu_{ei}\cos\theta. \end{aligned} \tag{42}$$

Now we deal with the collision of electrons with fast ions. We consider the case that $v_{te} \gg u_f$, where v_{te} and u_f are electron thermal velocity and fast ion beam velocity, respectively. In this case, the collision term can be approximated as

$$C(f_{em}, f_f) = \frac{m_e}{T_e}\nu_{ef}(v)v_{\|}u_{f\|}f_{em}, \tag{43}$$

where

$$\nu_{ef} = \frac{Z_f^2 n_f}{Z_{\rm eff}n_e}\nu_{ei}(v). \tag{44}$$

Using Eqs. (42) and (43), Eq. (38) is written as

$$v_{\|}\hat{\boldsymbol{b}}\cdot\nabla g - C^l(g) = \frac{m_e}{T_e}\frac{Z_f^2 n_f}{Z_{\rm eff}n_e}\nu_{ei}v_{\|}u_{f\|}f_{em} + f_{em}v\frac{I}{\Omega_e}\frac{\partial n_e}{\partial\psi}\frac{1}{n_e}\nu_{ei}\cos\theta \tag{45}$$

$$\Rightarrow v_{\|}\hat{\boldsymbol{b}}\cdot\nabla g - C^l(g) = f_{em}\nu_{ei}\frac{I v_{\|}}{\Omega_e}\left(\frac{\Omega_e}{I}\frac{m_e}{n_e T_e}\frac{Z_f^2 n_f}{Z_{\rm eff}}u_{f\|} + \frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\right) \tag{46}$$

Define $j_{f\|} = n_f Z_f e u_{f\|}$, then the above equation is written as

$$v_{\|}\hat{\boldsymbol{b}}\cdot\nabla g - C^l(g) = f_{em}\nu_{ei}\frac{I v_{\|}}{\Omega_e}\left(-\frac{1}{I}\frac{1}{n_e T_e}\frac{Z_f}{Z_{\rm eff}}j_{f\|}B + \frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\right) \tag{47}$$

$$\Rightarrow v_{\|}\hat{\boldsymbol{b}}\cdot\nabla g - C^l(g) = \nu_{ei}\frac{I}{\Omega_e}v_{\|}f_{em}\left(-\frac{1}{I p_e}\frac{Z_f}{Z_{\rm eff}}j_{f\|}B + \frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\right), \tag{48}$$

where $p_e \equiv n_e T_e$. Eq. (48) agrees with Eq. (8) in Lin-Liu's paper[5].

3 Sauter's theory of neoclassical bootstrap current coefficients

The trapped particles effect on beam driven current has close relation with the neoclassical bootstrap current coefficients. Before we discuss the trapped electrons effect on beam driven current, we should first understand the theory of bootstrap current. The following is a review of Sauter's theory of bootstrap current[6]. We start from the basic equation

$$v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla f_{e1}-C_e^l(f_{e1})=-\boldsymbol{v}_{\rm de}\cdot\nabla f_{em}-\frac{q_eE_{\parallel}}{m_e}\frac{\partial f_{em}}{\partial v}\cos\theta \tag{49}$$

$$\Rightarrow v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla f_{e1}-C_e^l(f_{e1})=-Iv_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla\left(\frac{v_{\parallel}}{\Omega_e}\right)f_{em}\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}+\frac{q_eE_{\parallel}}{T_e}v_{\parallel}f_{em}. \tag{50}$$

We want to determine the parallel current

$$j_{\parallel}=q_e\int f_{e1}v_{\parallel}d\Gamma, \tag{51}$$

where $d\Gamma$ is the volume element of velocity space. Instead of directly solving Eq. (50) to obtain the parallel current $j_{\parallel}$, it turns out that we can obtain $j_{\parallel}$ though the following way. First solve the adjoint equation

$$-v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla\chi_e-C_e^l(\chi_e)=q_ev_{\parallel}Bf_{em}. \tag{52}$$

Then $j_{\parallel}$ can be expressed as (proof of this is given in Sec. 4)

$$\langle j_{\parallel}B\rangle=\left\langle\int d\Gamma\frac{\chi_e}{f_{em}}S\right\rangle, \tag{53}$$

where S is the sum of all the source terms on the right-hand side of Eq. (50). We now calculate respectively the contribution of the every source term to the $j_{\parallel}$. First we consider the contribution of the parallel electrical field. For this term, Eq. (53) is written as

$$\begin{aligned}\langle j_{\parallel}B\rangle &= \left\langle\int d\Gamma\frac{\chi_e}{f_{em}}\frac{q_eE_{\parallel}}{T_e}v_{\parallel}f_{em}\right\rangle\\ &= \frac{q_e}{T_e}\left\langle E_{\parallel}\int d\Gamma\chi_e v_{\parallel}\right\rangle\\ &= \frac{q_e}{T_e}\left\langle E_{\parallel}B\frac{1}{B}\int d\Gamma\chi_e v_{\parallel}\right\rangle,\end{aligned} \tag{54}$$

Noting that $(\int d\Gamma\chi_ev_{\parallel})/B$ is a function of only ψ (this should be valid only for banana regime), i.e., it is independent of poloidal angle, thus this term can be taken out of the flux average, giving

$$\langle j_{\parallel}B\rangle=\frac{q_e}{T_e}\langle E_{\parallel}B\rangle\frac{1}{B}\int d\Gamma\chi_e v_{\parallel} \tag{55}$$

Define

$$\sigma_{\rm neo}=\frac{q_e}{T_e}\frac{1}{B}\int d\Gamma\chi_e v_{\parallel}, \tag{56}$$

then

$$\langle j_{\parallel}B\rangle=\sigma_{\rm neo}\langle E_{\parallel}B\rangle. \tag{57}$$

Next we calculate the contribution of the density gradient term to $j_{\parallel}$. In order to make the result be easily compared with Sauter's one, I adopt the method used in Sec. 2 to write f_{e1} in the form

$$f_{e1}=-\frac{Iv_{\parallel}}{\Omega_e}\frac{\partial f_{em}}{\partial\psi}+g_{e1}. \tag{58}$$

Following the same way as in Sec. 2, we can obtain an equation for g_{e1}

$$v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla g_{e1}-C^l(g_{e1})=\nu_{ei}\frac{I}{\Omega_e}v_{\parallel}f_{em}\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}. \tag{59}$$

Thus the parallel current contributed by g_{e1} is written as

$$\begin{aligned}\left\langle j_{g\parallel}B\right\rangle &= \left\langle \int d\Gamma\frac{\chi_e}{f_{em}}\left(\nu_{ei}\frac{I}{\Omega_e}v_{\parallel}f_{em}\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\right)\right\rangle \\ &= \left\langle \int d\Gamma\chi_e\nu_{ei}\frac{Iv_{\parallel}}{\Omega_e}\right\rangle\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}, \qquad (60)\end{aligned}$$

The parallel current contributed by the first term on the r.h.s of Eq. (58) is

$$\begin{aligned}j'_{\parallel} &= -q_e\int d\Gamma v_{\parallel}\frac{Iv_{\parallel}}{\Omega_e}\frac{\partial f_{em}}{\partial\psi} \\ &= -q_e\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\frac{I}{\Omega_e}\int d\Gamma v_{\parallel}^2 f_{em} \qquad (61)\end{aligned}$$

Using

$$\int d\Gamma v_{\parallel}^2 f_{em} = n_e\frac{T_e}{m_e},$$

Eq. (61) is written as

$$j'_{\parallel}B = -Ip_e\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}, \qquad (62)$$

where $p_e = n_e T_e$. Thus, using Eqs. (60) and (62), the total parallel current density is written as

$$\begin{aligned}\left\langle (j'_{\parallel}+j_{g\parallel})B\right\rangle &= -Ip_e\frac{1}{n_e}\frac{\partial n_e}{\partial\psi} + \left\langle \int d\Gamma\chi_e\nu_{ei}\frac{Iv_{\parallel}}{\Omega_e}\right\rangle\frac{1}{n_e}\frac{\partial n_e}{\partial\psi} \\ &= Ip_e\left[-1+\frac{1}{Ip_e}\left\langle \int d\Gamma\chi_e\nu_{ei}\frac{Iv_{\parallel}}{\Omega_e}\right\rangle\right]\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}, \qquad (63)\end{aligned}$$

Comparing Eq. (63) with Eq. (5) in Sauter's paper, we identify the $\mathcal{L}_{31}$ in Sauter's equation with the quantity in the bracket of Eq. (63), i.e.,

$$\mathcal{L}_{31} = -1+\frac{1}{Ip_e}\left\langle \int d\Gamma\chi_e\nu_{ei}\frac{Iv_{\parallel}}{\Omega_e}\right\rangle. \qquad (64)$$

Noting that $\nu_{ei} = Z_i\Gamma^{e/e}/v^3$ and $\Gamma^{e/e} = \nu_{e0}v_{te}^3$, we find that Eq. (64) agrees with Eq. (8) in Sauter's paper. (There is a difference of minus, please check.**)

4 Theory of the adjoint method

The perturbed distribution function satisfies the linearized Fokker-Planck equation,

$$v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla f_{e1} - C_e^l(f_{e1}) = S, \qquad (65)$$

where $\hat{\boldsymbol{b}}$ is the unit vector along the equilibrium magnetic field, $v_{\parallel}$ is the velocity component parallel to the magnetic field, $C_e^l(f_{e1})$ is the linearized collision operator, $C_e^l(f_{e1}) = C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C^{e/i}(f_{e1})$, S is a source term which is assumed to be known to us.

We want to determine the first moment of f_1,

$$j_{\parallel} = q_e\int f_{e1}v_{\parallel}d\Gamma, \qquad (66)$$

where $d\Gamma$ is volume element of velocity space. It turns out that we can obtain $j_{\parallel}$ through the following way. First solve the following equation (this equation will be called adjoint equation hereafter)

$$-v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla\chi - C_e^{l+}(\chi) = q_e v_{\parallel}B, \qquad (67)$$

where $C_e^{l+}(\chi) = C_e^l(f_{em}\chi)/f_{em}$. Then multiplying Eq. (67) by f_{e1}, one gets

$$-f_{e1}v_{\parallel}\hat{\boldsymbol{b}}\cdot\nabla\chi - f_{e1}C_e^{l+}(\chi) = q_e f_{e1}v_{\parallel}B \qquad (68)$$

Integrate both sides of the above equation in velocity space, one gets

$$\Longrightarrow \int d\Gamma \Big[-f_{e1} v_{\parallel} \hat{b} \cdot \nabla \chi - f_{e1} C_e^{l+}(\chi) \Big] = j_{\parallel} B \tag{69}$$

Flux averaging both sides of the above equation gives

$$\left\langle \int d\Gamma \Big[-f_{e1} v_{\parallel} \hat{b} \cdot \nabla \chi - f_{e1} C_e^{l+}(\chi) \Big] \right\rangle = \langle j_{\parallel} B \rangle \tag{70}$$

Now we need to use the most important properties of the operator $v_{\parallel} \hat{b} \cdot \nabla$ and C_e^{l+}, i.e., adjoint properties,

$$\left\langle \int d\Gamma f v_{\parallel} \hat{\boldsymbol{b}} \cdot \nabla g \right\rangle = -\left\langle \int d\Gamma g v_{\parallel} \hat{\boldsymbol{b}} \cdot \nabla f \right\rangle, \tag{71}$$

and

$$\int d\Gamma f_{e1} C_e^{l+}(\chi) = \int d\Gamma \chi C_e^l(f_{e1}) \tag{72}$$

[Refer to another note for the proof of Eq. (71).] Using the above two properties, Eq. (70) is written as

$$\left\langle \int d\Gamma \chi \Big[v_{\parallel} \hat{b} \cdot \nabla f_1 - C_e^l(f_1) \Big] \right\rangle = \langle j_{\parallel} B \rangle \tag{73}$$

Using Eq. (65) to rewrite the term in the bracket of the above equation, we obtain

$$\langle j_{\parallel} B \rangle = \left\langle \int d\Gamma \chi S \right\rangle. \tag{74}$$

Eq. (74) is the desired formula for calculating $j_{\parallel}$.

Note that Eq. (67) can also be written as

$$-v_{\parallel} \hat{\boldsymbol{b}} \cdot \nabla \chi - C_e^l(\chi f_{em}) / f_{em} = q_e v_{\parallel} B, \tag{75}$$

$$\Rightarrow -v_{\parallel} f_{em} \hat{\boldsymbol{b}} \cdot \nabla \chi - C_e^l(\chi f_{em}) = q_e v_{\parallel} B f_{em}, \tag{76}$$

Since f_{em} is independent of the poloidal angle, f_{em} can be moved into the operator $\hat{\boldsymbol{b}} \cdot \nabla()$, giving

$$\Rightarrow -v_{\parallel} \hat{\boldsymbol{b}} \cdot \nabla(\chi f_{em}) - C_e^l(\chi f_{em}) = q_e v_{\parallel} B f_{em}. \tag{77}$$

If we define $\chi' = \chi f_{em}$, then the above equation is written as

$$-v_{\parallel} \hat{\boldsymbol{b}} \cdot \nabla(\chi') - C_e^l(\chi') = q_e v_{\parallel} B f_{em}, \tag{78}$$

This form of the adjoint equation is almost identical to its original equation (65), the minor difference being that an additional minus mark appears on the first term and the inhomogenous term on the right-hand side is replaced with $q_e v_{\parallel} B f_{em}$. In terms of χ', Eq. (74) is written as

$$\langle j_{\parallel} B \rangle = \left\langle \int d\Gamma \frac{\chi'}{f_{em}} S \right\rangle. \tag{79}$$

Sauter[6] adopted Eqs. (78) and (79) in the calculation of bootstrap current.

5 Beam driven current in uniform plasmas

In this section we consider the calculation of the beam-driven current in uniform plasmas[2]. We consider the case that a neutral beam is injected to plasma, ionized to become fast ions beam, and the steady state has been reached. If the steady-state beam distribution function, f_b, is known, the steady-state perturbed electron distribution function can be determined from the following equation

$$C_e(f_{e1}) = -C(f_{em}, f_b), \tag{80}$$

where $C_e(f_{e1}) \equiv C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C(f_{e1}, f_{im})$. We are interested in parallel (to magnetic field) current

$$j_{\|} = q_b \int v_{\|} f_b(\boldsymbol{v}) d\boldsymbol{v} - e \int v_{\|} f_{e1}(\boldsymbol{v}) d\boldsymbol{v}, \tag{81}$$

Using the property of the collision operators that the Legendre harmonics are the eigen-functions of the collision operators,

$$C(f_{em}, f(v) P_l(\cos\theta)) = g(l, v) P_l(\cos\theta) \tag{82}$$

$$C(f(v) P_l(\cos\theta), f_{em}) = h(l, v) P_l(\cos\theta) \tag{83}$$

we get

$$C_e(f(v) P_l(\cos\theta)) = y(l, v) P_l(\cos\theta). \tag{84}$$

Eqs. (80) and (84) indicate that it is only the first Legendre harmonic of f_b that can generate the first Legendre harmonic perturbation in electron distribution. Further note that the parallel electron current is the inner product of $P_1(\cos\theta)$ and f_{e1}, and Legendre harmonics are orthogonal to each other, thus only the first Legendre harmonic of f_{e1} can contribute to the parallel electron current. Considering the above results, in order to calculate the parallel current in Eq. (81) [Note $v_{\|} = v P_1(\cos\theta)$], it is sufficient to consider only the $l = 1$ Legendre harmonics of the beam distribution function in Eq. (80), $f_b^1(v)\cos\theta$, here

$$f_b^1(v) = \frac{3}{2} \int_0^{\pi} f_b(v, \theta) P_1(\cos\theta) \sin\theta d\theta \tag{85}$$

Before treating arbitrary $f_b(v, \theta)$, we first consider the case that the velocity dependence of $f_b(v, \theta)$ is given by the Dirac delta function

$$f_b(v, \theta) = \delta(v - v_b) g(v, \theta), \tag{86}$$

which is equivalent to

$$f_b(v, \theta) = \delta(v - v_b) g(v_b, \theta). \tag{87}$$

Then $n_b u_{b\|}$ can be written as

$$\begin{aligned} n_b u_{b\|} &\equiv \int f_b(v, \theta) v_{\|} d\boldsymbol{v} \\ &= 2\pi \int_0^{\infty} \int_0^{\pi} \delta(v - v_b) g(v_b, \theta) v^3 \cos\theta \sin\theta dv d\theta \\ &= 2\pi \int_0^{\pi} g(v_b, \theta) v_b^3 \cos\theta \sin\theta d\theta. \end{aligned} \tag{88}$$

This gives the relation of $u_{b\|}$ and v_b. Using this result, $f_b^1(v)$ in Eq. (85) is written as

$$\begin{aligned} f_b^1(v) &= \frac{3}{2} \int_0^{\pi} \delta(v - v_b) g(v_b, \theta) P_1(\cos\theta) \sin\theta d\theta \\ &= \frac{3}{2} \frac{\delta(v - v_b)}{v_b^3} \int_0^{\pi} g(v_b, \theta) v_b^3 \cos\theta \sin\theta d\theta \\ &= \frac{3}{2} \frac{\delta(v - v_b)}{v_b^3} \frac{n_b u_{b\|}}{2\pi}, \end{aligned} \tag{89}$$

which agrees with Hirshman's result,

$$f_b^1(v) = \frac{3 n_b u_{b\|} \delta(v - v_b)}{4\pi v_b^3}. \tag{90}$$

Using the property of the Dirac delta function, $\delta(a\xi) = \delta(\xi)/a$, Eq. (90) can be written as

$$f_b^1(v) = \frac{3 n_b u_{\|b} \delta(\bar{v} - \bar{v}_b)}{4\pi v_b^3 v_{te}}, \tag{91}$$

where $\bar{v} = v/v_{te}$, $\bar{v}_b = v_b/v_{te}$. Note that the dimension of $f_b^1(v)$ is correct, i.e, $f_b^1 \propto n/v^3$.

In the following, we calculate the collision term $C(f_{em},\, f_b^1(v)\cos\theta)$. According to Eq. (34) in Karney's paper[14], we have

$$
\begin{aligned}
\frac{C(f_{am},\, f_b^1(v)\cos\theta)}{f_{am}\cos\theta} &= \frac{-\nabla\cdot\boldsymbol{S}_c^{a/b}}{f_{am}\cos\theta}\\
&= \frac{4\pi\Gamma^{a/b}}{n_b}\Bigg[f_b^1(v)+\int_0^v\frac{v'^2}{v_{ta}^2}\bigg(\frac{v'^3}{5v_{ta}^2v^2}-\frac{v'}{3v^2}\bigg)f_b^1(v')dv'\\
&+\int_v^\infty\frac{v'^2}{v_{ta}^2}\bigg(\frac{v^3}{5v_{ta}^2v'^2}-\frac{v}{3v'^2}\bigg)f_b^1(v')dv'\Bigg]
\end{aligned}
$$

Substituting $f_b^1(v)$ in Eq. (91) into the above equation gives

$$
\begin{aligned}
\frac{C(f_{em},\, f_b^1(v)\cos\theta)}{f_{em}\cos\theta} &= \Gamma^{e/b}\frac{u_{\|b}}{v_b^3v_{te}}\Bigg[3\delta(\overline{v}-\overline{v}_b)+\int_0^{\overline{v}}\overline{v}'^2\bigg(\frac{\overline{v}'^3}{5\overline{v}^2}-\frac{\overline{v}'}{3\overline{v}^2}\bigg)3\delta(\overline{v}'-\overline{v}_b)d\,\overline{v}'\\
&+\int_{\overline{v}}^\infty\bigg(\frac{\overline{v}^3}{5}-\frac{\overline{v}}{3}\bigg)3\delta(\overline{v}'-\overline{v}_b)d\overline{v}'\Bigg]
\end{aligned} \tag{92}
$$

It follows that

$$
\frac{C(f_{em},\, f_b^1(v)\cos\theta)}{f_{em}\cos\theta}=\begin{cases}\dfrac{\Gamma^{e/b}}{v_{te}^3}\dfrac{u_{\|b}}{v_{te}}\dfrac{1}{\overline{v}_b}\bigg(\dfrac{3\overline{v}_b^3}{5\overline{v}^2}-\dfrac{\overline{v}_b}{\overline{v}^2}\bigg) & \text{For } \overline{v}_b<\overline{v}\\ \dfrac{\Gamma^{e/b}}{v_{te}^3}\dfrac{u_{\|b}}{v_{te}}\dfrac{1}{\overline{v}_b^3}\bigg(\dfrac{3\overline{v}^3}{5}-\overline{v}\bigg) & \text{For } \overline{v}_b>\overline{v}\\ \dfrac{\Gamma^{a/b}}{v_{te}^3}\dfrac{u_{\|b}}{v_{te}}\dfrac{1}{\overline{v}_b^3}3\delta(\overline{v}-\overline{v}_b) & \text{For } \overline{v}_b=\overline{v}\end{cases} \tag{93}
$$

where

$$
\Gamma^{a/b}=\frac{n_bq_a^2q_b^2}{4\pi\epsilon_0^2m_a^2}\ln\Lambda^{a/b},\ \Gamma^{e/b}=\frac{n_be^2q_b^2}{4\pi\epsilon_0^2m_e^2}\ln\Lambda^{e/b},\ \Gamma^{e/e}=\frac{n_ee^4}{4\pi\epsilon_0^2m_e^2}\ln\Lambda^{e/e} \tag{94}
$$

Define the effective charge number of beam ions, $\overline{Z}_b$, as

$$
\overline{Z}_b\equiv\frac{n_bq_b^2}{n_ee^2}\frac{\ln\Lambda^{e/b}}{\ln\Lambda^{e/e}}, \tag{95}
$$

then $\Gamma^{e/b}=\overline{Z}_b\Gamma^{e/e}$ and Eq. (93) is written as

$$
\frac{C(f_{em},\, f_b^1(v)\cos\theta)}{f_{em}\cos\theta}=\begin{cases}\overline{Z}_b\nu_{e0}\dfrac{u_{\|b}}{v_{te}}\dfrac{1}{\overline{v}^2}\bigg(\dfrac{3}{5}\overline{v}_b^2-1\bigg) & \text{For } \overline{v}_b<\overline{v}\\ \overline{Z}_b\nu_{e0}\dfrac{u_{\|b}}{v_{te}}\dfrac{\overline{v}}{\overline{v}_b^3}\bigg(\dfrac{3}{5}\overline{v}^2-1\bigg) & \text{For } \overline{v}_b>\overline{v}\\ \overline{Z}_b\nu_{e0}\dfrac{u_{\|b}}{v_{te}}\dfrac{1}{\overline{v}_b^3}3\delta(\overline{v}-\overline{v}_b) & \text{For } \overline{v}_b=\overline{v}\end{cases} \tag{96}
$$

where $\nu_{e0}=\Gamma^{e/e}/v_{te}^3$. Hirshman's result[1] of the above collision term is given by

$$
C(f_{em},\, f_b^1(v)\cos\theta)=\begin{cases}\dfrac{2v_\|u_{\|b}}{v_e^2}\dfrac{n_bq_b^2}{n_ee^2}\nu_ef_{em}\dfrac{1}{\overline{v}^3}(1+\dfrac{6}{5}\overline{v}_b^2) & \text{For } \overline{v}_b<\overline{v}\\ \dfrac{2v_\|u_{\|b}}{v_e^2}\dfrac{n_bq_b^2}{n_ee^2}\nu_ef_{em}\dfrac{1}{\overline{v}_b^3}\bigg(\dfrac{6}{5}\overline{v}^2-2\bigg) & \text{For } \overline{v}_b>\overline{v}\end{cases} \tag{97}
$$

where $v_e=\sqrt{2T_e/m_e}$, $\nu_e=\Gamma^{e/e}/v_e^3,\overline{v}=v/v_e,\overline{v}_b=v_b/v_e$. Converted to my normalization, Eq. (97) takes the form,

$$
\frac{C(f_{em},\, f_b^1(v)\cos\theta)}{f_{em}\cos\theta}=\begin{cases}\overline{Z}_b\dfrac{\nu_{e0}}{2\sqrt{2}}\dfrac{2u_{\|b}}{\sqrt{2}\,v_{te}}\dfrac{1}{\overline{v}^2/2}(\dfrac{6}{5}\overline{v}_b^2/2+1) & \text{For } \overline{v}_b<\overline{v}\\ \overline{Z}_b\dfrac{\nu_{e0}}{2\sqrt{2}}\dfrac{4u_{\|b}}{\sqrt{2}v_{te}}\dfrac{\overline{v}/\sqrt{2}}{\overline{v}_b^3/2\sqrt{2}}\bigg(\dfrac{3}{5}\overline{v}^2/2-1\bigg) & \text{For } \overline{v}_b>\overline{v}\end{cases} \tag{98}
$$

which can be simplified to

$$\frac{C(f_{em}, f_b^1(v)\cos\theta)}{f_{em}\cos\theta} = \begin{cases} \overline{Z}_b \nu_{e0} \frac{u_{\parallel b}}{v_{te}} \frac{1}{\overline{v}^2} (\frac{3}{5}\overline{v}_b^2 + 1) & \text{For } \overline{v}_b < \overline{v} \\ \overline{Z}_b \nu_{e0} \frac{u_{\parallel b}}{v_{te}} \frac{\overline{v}}{\overline{v}_b^3} \left(\frac{3}{5}\overline{v}^2 - 2\right) & \text{For } \overline{v}_b > \overline{v} \end{cases} \tag{99}$$

Note that Eq. (99) is different from my results, Eq. (96). Using Eq. (99) in my code, I can reproduce Hirshman's results [Figs. 1 and 2, and the analytic expression Eq. (24c) in his paper]. My question is why Eq. (99) differs from Eq.(96) for $\overline{v}_b < \overline{v}$ and $\overline{v}_b > \overline{v}$, and why Eq. (99) does not involve the Dirac delta function at $\overline{v} = \overline{v}_b$. Using expression Eq. (96), I can not reproduce Hirshman's results.

5.1 Collision term of electron with fast ions in the limit $v_b \ll v_{te}$

In dealing with the electron shielding current in NBCD in the case that $v_b \ll v_{te}$, the collision term of electrons with fast ions is approximated as (Eq. (6) in Ref. [15] or the last term of Eq. (5) in Ref. [5])

$$C^{e/f}(f_{em}) = \frac{m_e}{T_e} \nu_{ef} v_\parallel u_{f\parallel} f_{em}, \tag{100}$$

which can be rewritten as

$$\begin{aligned} C^{e/f}(f_{em}) &= \frac{1}{v_{te}^2} \frac{Z_f^2 n_f \nu_{ei}}{Z_{\text{eff}} n_e} v_\parallel u_{f\parallel} f_{em} \\ &= \frac{1}{v_{te}^2} \frac{Z_f^2 n_f \Gamma^{e/e}}{v^3 n_e} v_\parallel u_{f\parallel} f_{em} \\ &= \frac{Z_f^2 n_f}{n_e} \frac{\Gamma^{e/e}}{v^3} \frac{v_\parallel u_{f\parallel}}{v_{te}^2} f_{em} \end{aligned} \tag{101}$$

I now prove Eq. (101) by using Eq. (99). According to Eq. (99), in the limit that $v_b \ll v_{te}$, i.e., $\overline{v}_b \sim 0$, we have

$$\begin{aligned} C(f_{em}, f_b^1(v)\cos\theta) &= \overline{Z}_b \nu_{e0} \frac{u_{\parallel b}}{v_{te}} \frac{1}{\overline{v}^2} (\frac{3}{5}\overline{v}_b^2 + 1) f_{em}\cos\theta \\ &\approx \overline{Z}_b \nu_{e0} \frac{u_{\parallel b}}{v_{te}} \frac{1}{\overline{v}^2} f_{em}\cos\theta \\ &= \overline{Z}_b \frac{\Gamma^{e/e}}{v_{te}^3} \frac{u_{\parallel b}}{v_{te}} \frac{1}{\overline{v}^2} f_{em}\cos\theta \\ &= \overline{Z}_b \frac{\Gamma^{e/e}}{v_{te}^3} \frac{u_{\parallel b}}{v_{te}} \frac{v_{te}^2}{v^2} f_{em}\cos\theta \\ &= \overline{Z}_b \frac{\Gamma^{e/e}}{v^3} \frac{u_{\parallel b}}{v_{te}} \frac{v_{te}^2}{v^2} \frac{v^3}{v_{te}^3} f_{em}\cos\theta \\ &= \overline{Z}_b \frac{\Gamma^{e/e}}{v^3} \frac{u_{\parallel b}}{v_{te}} \frac{v}{v_{te}} f_{em}\cos\theta \\ &= \overline{Z}_b \frac{\Gamma^{e/e}}{v^3} \frac{u_{\parallel b}}{v_{te}} \frac{v_\parallel}{v_{te}} f_{em} \end{aligned} \tag{102}$$

Using

$$\overline{Z}_b \equiv \frac{n_b q_b^2}{n_e e^2} \frac{\ln\Lambda^{e/b}}{\ln\Lambda^{e/e}} = \frac{n_b q_b^2}{n_e e^2} \approx \frac{n_f Z_f^2 e^2}{n_e e^2} = \frac{n_f Z_f^2}{n_e}, \tag{103}$$

in Eq. (102), we obtain

$$C(f_{em}, f_b^1(v)\cos\theta) = \frac{n_f Z_f^2}{n_e} \frac{\Gamma^{e/e}}{v^3} \frac{u_{\parallel b}}{v_{te}} \frac{v_\parallel}{v_{te}} f_{em}, \tag{104}$$

which agrees with Eq. (101).

6 Adjoint Problem of Eq. (80)

Because of the complexity of the non-homogeneous term $C(f_{em}, f_b)$ in Eq. (80), the direct solution of this equation is usually difficult. We consider a simpler problem,

$$C_e(f_{em}\chi(\boldsymbol{v})) = \nu_{e0}\frac{v_\parallel}{v_{te}}f_{em} \tag{105}$$

whose non-homogeneous term is simpler than $C(f_{em}, f_b)$. In the following, we will prove that if the solution to Eq. (105) is known, the parallel current can be easily obtained without solving the more complicated equation Eq. (80).

Using the self-adjoint property of the linearized collision operator C_e,

$$\int \phi C_e(f_{em}\chi)d\boldsymbol{v} = \int \chi C_e(f_{em}\phi)d\boldsymbol{v} \tag{106}$$

(The proof of this property is given in another note.) the parallel current can be written

$$\begin{aligned}
j_{e\parallel} &\equiv q_e\int v_\parallel f_{e1}d\boldsymbol{v}\\
&= q_e\int v_\parallel f_{em}\frac{f_{e1}}{f_{em}}d\boldsymbol{v}\\
&= q_e\frac{v_{te}}{\nu_{e0}}\int C_e(f_{em}\chi)\frac{f_{e1}}{f_{em}}d\boldsymbol{v}\\
&= q_e\frac{v_{te}}{\nu_{e0}}\int \chi C_e(f_{e1})d\boldsymbol{v}\\
&= -q_e\frac{v_{te}}{\nu_{e0}}\int \chi C(f_{em}, f_b)\,d\boldsymbol{v}
\end{aligned} \tag{107}$$

Therefore, the parallel current only involves the velocity integration of χ and $C(f_{em}, f_b)$. The expression of $C(f_{em}, f_b)$ is given by Eq. (93) while χ is known from the solution to adjoint equation Eq. (105). Thus the parallel current is determined without solving Eq. (80).

7 Beam current ratio

$$\begin{aligned}
F &\equiv 1+\frac{j_{e\parallel}}{q_b n_b u_{\parallel b}}\\
&= 1+\frac{e\frac{v_{te}}{\nu_{e0}}\int \chi C(f_{em}, f_b)\,d\boldsymbol{v}}{q_b n_b u_{\parallel b}}\\
&= 1+\frac{e v_{te}\int \chi\,\overline{C}\,(f_{em}, f_b) f_{em}\cos\theta\, d\boldsymbol{v}}{\nu_{e0} q_b n_b u_{\parallel b}}\\
&= 1+\frac{e v_{te}\int \chi^1\,\overline{C}\,(f_{em}, f_b) f_{em}\cos^2\theta\, d\boldsymbol{v}}{\nu_{e0} q_b n_b u_{\parallel b}}\\
&= 1+\frac{4\pi}{3}\frac{e v_{te}\int_0^\infty \chi^1\,\overline{C}\,(f_{em}, f_b) f_{em} v^2 dv}{\nu_{e0} q_b n_b u_{\parallel b}}\\
&= 1+\frac{4\pi}{3}\frac{e v_{te} f_0\int_0^\infty \chi^1\,\overline{C}\,(f_{em}, f_b) \overline{f}_{em} v^2 dv}{\nu_{e0} q_b n_b u_{\parallel b}}\\
&= 1+\frac{4\pi}{3}\frac{e n_e}{q_b n_b}\frac{v_{te}}{u_{\parallel b}}\frac{1}{\nu_{e0}}\int_0^\infty \chi^1\,\overline{C}\,(f_{em}, f_b)\overline{f}_{em}\overline{v}^2 d\,\overline{v}
\end{aligned}$$

where

$$\chi(\boldsymbol{v}) = \chi^1(v)\cos\theta$$

$$\overline{C}\,(f_{em}, f_b) = \frac{C(f_{em}, f_b^1(v)\cos\theta)}{f_{em}\cos\theta}$$

$$f_{em} = \overline{f}_{em} f_0,\quad f_0 = \frac{n_e}{v_{te}^3}$$

Using

$$\overline{C}\,(f_{em}, f_b) = \begin{cases} \overline{Z}_b \nu_{e0} \dfrac{u_{\parallel b}}{v_{te}} \left(\dfrac{3\overline{v}_b^2}{5\overline{v}^2} - \dfrac{1}{\overline{v}^2} \right) & \text{For } \overline{v}_b < \overline{v} \\ \overline{Z}_b \nu_{e0} \dfrac{u_{\parallel b}}{v_{te}} \dfrac{1}{\overline{v}_b^3} \left(\dfrac{3\overline{v}^3}{5} - \overline{v} \right) & \text{For } \overline{v}_b > \overline{v} \\ \overline{Z}_b \nu_{e0} \dfrac{u_{\parallel b}}{v_{te}} \dfrac{1}{\overline{v}_b^3} 3\delta(\overline{v} - \overline{v}_b) & \text{For } \overline{v}_b = \overline{v} \end{cases} \tag{108}$$

one gets

$$\begin{aligned} F &= 1 + \frac{4\pi}{3} \frac{e n_e}{q_b n_b} \frac{v_{te}}{u_{\parallel b}} \frac{1}{\nu_{e0}} \int_0^\infty \chi^1 \,\overline{C}\,(f_{em}, f_b) \overline{f}_{em} \overline{v}^2 d\,\overline{v} \\ &= 1 + \frac{4\pi}{3} \frac{e n_e}{q_b n_b} \overline{Z}_b \\ &\times \left[\chi^1(\overline{v}_b) \frac{3}{\overline{v}_b} \overline{f}_{em}(\overline{v}_b) + \int_0^{\overline{v}_b} \chi^1 \frac{1}{\overline{v}_b^3} \left(\frac{3\overline{v}^3}{5} - \overline{v} \right) \overline{f}_{em} \overline{v}^2 d\,\overline{v} + \int_{\overline{v}_b}^\infty \chi^1 \left(\frac{3\overline{v}_b^2}{5\overline{v}^2} - \frac{1}{\overline{v}^2} \right) \overline{f}_{em} \overline{v}^2 d\,\overline{v} \right] \end{aligned}$$

Define $Z_b \equiv q_b/e$, which is the charge number of the fast ions, then

$$\overline{Z}_b = Z_b \frac{n_b q_b}{n_e e} \tag{109}$$

The expression of F reduces to

$$\begin{aligned} F &= 1 + \frac{4\pi}{3} Z_b \\ &\times \left[\chi^1(\overline{v}_b) \frac{3}{\overline{v}_b} \overline{f}_{em}(\overline{v}_b) + \frac{1}{\overline{v}_b^3} \int_0^{\overline{v}_b} \chi^1 \overline{f}_{em} \left(\frac{3\overline{v}^5}{5} - \overline{v}^3 \right) d\,\overline{v} + \left(\frac{3\overline{v}_b^2}{5} - 1 \right) \int_{\overline{v}_b}^\infty \chi^1 \overline{f}_{em} d\,\overline{v} \right] \end{aligned} \tag{110}$$

8 Results

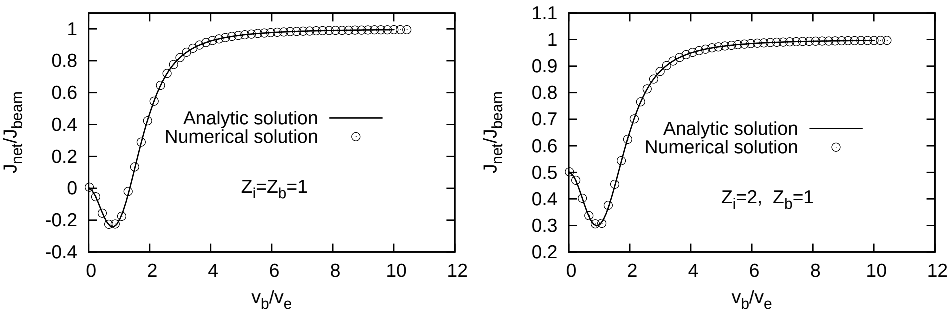

Figure 3. The ratio of net current to beam current as a function of beam velocity v_b. Here $Z_i = n_i q_i^2 / n_e e^2$. $Z_b = q_b / e$.

As mentioned above, the results in Fig. 3 is obtained by using Eq. (99) as the collision term. In this case, the expression of F takes the form,

$$\begin{aligned} F &= 1 + \frac{4\pi}{3} Z_b \\ &\times \left[\frac{1}{\overline{v}_b^3} \int_0^{\overline{v}_b} \chi^1 \overline{f}_{em} \left(\frac{3\overline{v}^5}{5} - 2\overline{v}^3 \right) d\,\overline{v} + \left(\frac{3\overline{v}_b^2}{5} + 1 \right) \int_{\overline{v}_b}^\infty \chi^1 \overline{f}_{em} d\,\overline{v} \right] \end{aligned}$$

The analytic solution is from Eq. (24c) in Hirshman's paper[2]. In this case the numerical solution agree well with the analytic solution as shown in Fig. 3. However, if we use Eq. (110) to calculate F (i.e., Eq. (96) is used as collision term), the results does not agree with the analytic solution. The results are plotted in Fig. 4.

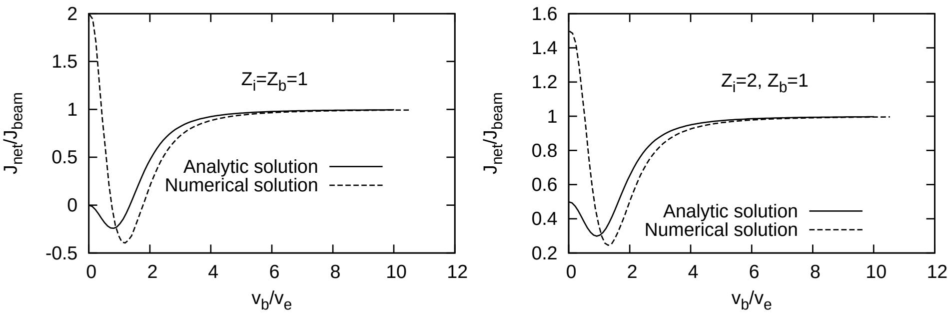

Figure 4. F calculated using Eq. (110), which does not agree with Hirshman's analytic solution in the region of small values of v_b.

Why Eq. (110) can not reproduce the correct results is still a mystery to me.

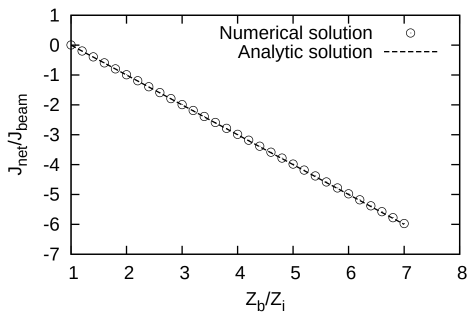

Figure 5. The ratio of the net current to beam current as a function of Z_b/Z_i. The analytic solution is given by $F = 1 - \frac{Z_b}{Z_i}\left(1 + \frac{6}{5}\left(\frac{v_b}{v_e}\right)^2\right)$ which is valid only for $v_b/v_e \ll 1$. So in this case the value of v_b/v_e is chosen to be small, $v_b/v_e = 0.01$.

9 To prove $\boldsymbol{v_d}\cdot\boldsymbol{\nabla}\boldsymbol{\psi} = -\boldsymbol{I}\boldsymbol{v}_\parallel\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla}\left(\frac{\boldsymbol{v}_\parallel}{\boldsymbol{\Omega}}\right)$

We start from Eq. (4.6.3) in Wesson's book[13]

$$\boldsymbol{v}_d = \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_c}\frac{\boldsymbol{B}\times\nabla B}{B^2}, \tag{111}$$

where $\omega_c = Bq/m$ (here q is the charge of particle, including its sign). Then we write

$$\begin{aligned}\boldsymbol{v}_d\cdot\nabla\psi &= \frac{v_\parallel^2 + \frac{1}{2}v_\perp^2}{\omega_c}\frac{\boldsymbol{B}\times\nabla B}{B^2}\cdot\nabla\psi \\ &= -\frac{v_\parallel^2 + \mu B/m}{\omega_c}\nabla B\cdot\frac{\boldsymbol{B}\times\nabla\psi}{B^2},\end{aligned} \tag{112}$$

where $\mu = mv_\perp^2/(2B)$. Eq. (4.6.7) in Wesson's book is (I have checked this equation)

$$\left(v_\parallel^2 + \frac{\mu}{m}B\right)\nabla B = -v_\parallel B^2\nabla\left(\frac{v_\parallel}{B}\right). \tag{113}$$

Using this in Eq. (112), we obtain

$$\boldsymbol{v}_d\cdot\nabla\psi = \frac{v_\parallel}{\omega_c}(\boldsymbol{B}\times\nabla\psi)\cdot\nabla\left(\frac{v_\parallel}{B}\right). \tag{114}$$

Writing

$$\boldsymbol{B} = B_\phi \boldsymbol{e}_\phi + \frac{1}{R}(\nabla\psi \times \boldsymbol{e}_\phi), \tag{115}$$

we obtain

$$\begin{aligned}\boldsymbol{B}\times\nabla\psi &= B_\phi \boldsymbol{e}_\phi \times \nabla\psi + \frac{1}{R}(\nabla\psi\times\boldsymbol{e}_\phi)\times\nabla\psi \\ &= B_\phi \boldsymbol{e}_\phi \times \nabla\psi + \frac{1}{R}|\nabla\psi|^2 \boldsymbol{e}_\phi,\end{aligned} \tag{116}$$

where use has been made of $\boldsymbol{e}_\phi \cdot \nabla\psi = 0$. Then Eq. (114) is written as

$$\begin{aligned}\boldsymbol{v}_d\cdot\nabla\psi &= \frac{v_\parallel}{\omega_c}\left(B_\phi \boldsymbol{e}_\phi \times \nabla\psi + \frac{1}{R}|\nabla\psi|^2 \boldsymbol{e}_\phi\right)\cdot\nabla\left(\frac{v_\parallel}{B}\right) \\ &= \frac{v_\parallel}{\omega_c}(B_\phi \boldsymbol{e}_\phi \times \nabla\psi)\cdot\nabla\left(\frac{v_\parallel}{B}\right),\end{aligned} \tag{117}$$

where the second equality is due to

$$\boldsymbol{e}_\phi \cdot \nabla\left(\frac{v_\parallel}{B}\right) = 0, \tag{118}$$

since we are considering toroidal symmetric case. Eq. (117) is further written as

$$\boldsymbol{v}_d\cdot\nabla\psi = -\frac{v_\parallel}{\omega_c} I \left(\frac{\nabla\psi\times\boldsymbol{e}_\phi}{R}\right)\cdot\nabla\left(\frac{v_\parallel}{B}\right), \tag{119}$$

where $I = B_\phi R$. Using Eq. (115) in the above equation gives

$$\boldsymbol{v}_d\cdot\nabla\psi = -\frac{v_\parallel}{\omega_c} I (\boldsymbol{B} - B_\phi \boldsymbol{e}_\phi)\cdot\nabla\left(\frac{v_\parallel}{B}\right), \tag{120}$$

Because of Eq. (118), the above equation is reduced to

$$\boldsymbol{v}_d\cdot\nabla\psi = -\frac{v_\parallel}{\omega_c} I \boldsymbol{B}\cdot\nabla\left(\frac{v_\parallel}{B}\right), \tag{121}$$

which agrees with Eq. (4.6.9) in Wesson's book[13].

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Bibliography

[1] T. Ohkawa. New methods of driving plasma current in fusion devices. *Nuclear Fusion*, 10(2):185, 1970.

[2] S. P. Hirshman. Classical collisional theory of beam-driven plasma currents. *Phys. Fluids*, 23(6):1238–1243, 1980.

[3] M. Taguchi. Approximate expression for beam driven current in tokamak plasmas. *Nuclear Fusion*, 32(1):143, 1992.

[4] D. F. H. Start and J. G. Cordey. Beam-induced currents in toroidal plasmas of arbitrary aspect ratio. *Phys. Fluids*, 23(7):1477–1478, 1980.

[5] Y. R. Lin-Liu and F. L. Hinton. Trapped electron correction to beam driven current in general tokamak equilibria. *Phys. Plasmas*, 4(11):4179–4181, 1997.

[6] O. Sauter, C. Angioni, and Y. R. Lin-Liu. Neoclassical conductivity and bootstrap current formulas for general axisymmetric equilibria and arbitrary collisionality regime. *Phys. Plasmas*, 6(7):2834–2839, 1999.

[7] H. E. St. John, T. S. Taylor, Y. R. Lin-Liu, and A. D. Turnbull. In *Conf. Proc., 15th Int. Conf. on Plasma Phys. and Control. Nucl. Fusion Research*, volume 3, page 603, Seville, Spain, 1994. IAEA, Vienna, 1995.

[8] R.J Goldston, D.C McCune, H.H Towner, S.L Davis, R.J Hawryluk, and G.L Schmidt. New techniques for calculating heat and particle source rates due to neutral beam injection in axisymmetric tokamaks. *J. of Comput. Phys.*, 43(1):61 – 78, 1981.

[9] T. Oikawa, J.M. Park, A.R. Polevoi, M. Schneider, G. Giruzzi, M. Murakami, K. Tani, A.C.C. Sips, C. Kesse, W. Houlberg, S. Konovalov, K. Hamamatsu, V. Basiuk, A. Pankin, D. McCune, R. Budny, Y-S. Na, I.Voitsekhovich, and S. Suzuki. Benchmarking of neutral beam current drive codes as a basis for the integrated modeling for iter. *22nd IAEA Fusion Energy Conference*, IT/P6-5, 2008.

[10] S.P. Hirshman and D.J. Sigmar. Neoclassical transport of impurities in tokamak plasmas. *Nuclear Fusion*, 21(9):1079, 1981.

[11] Steven P. Hirshman. Neoclassical current in a toroidally-confined multispecies plasma. *Phys. Fluids*, 21(8):1295–1301, 1978.

[12] M. Kraus. *TRANSP Current Drive Algorithms*, http://w3.pppl.gov/transp/papers/.

[13] John Wesson. *Tokamaks*. Oxford University Press, 2004.

[14] Charles F. F. Karney. Fokker-planck and quasilinear codes. *Comp. Phys. Rep.*, 4:183–244, 1986.

[15] Y. J. Hu, Y. M. Hu, and Y. R. Lin-Liu. Electron shielding current in neutral beam current drive in general tokamak equilibria and arbitrary collisionality regime. *Physics of Plasmas*, 19(3):034505, 2012.