Fokker-Planck collision operator

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1 Non-relativistic Fokker-Planck collision operator

The influence of Coulomb collisions on the time evolution of distribution function can be written as

$$\frac{\partial f_a}{\partial t} = -\nabla\cdot\boldsymbol{S}_c = -\nabla\cdot\sum_b \boldsymbol{S}_c^{a/b}, \tag{1}$$

where $\nabla \equiv \partial/\partial\boldsymbol{v}$ is the gradiant operator in velocity space, $\boldsymbol{S}_c^{a/b}$ is the collision flux, which takes the Landau integral form:

$$\boldsymbol{S}_c^{a/b} = \frac{c_{ab}}{m_a}\int d\boldsymbol{v}'\left(\frac{\boldsymbol{I}}{s} - \frac{\boldsymbol{s}\boldsymbol{s}}{s^3}\right)\cdot\left[\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') - \frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v})\right], \tag{2}$$

where $\boldsymbol{s} = \boldsymbol{v} - \boldsymbol{v}'$, $\nabla' \equiv \partial/\partial\boldsymbol{v}'$, $c_{ab} = q_a^2 q_b^2 \ln\Lambda^{a/b}/8\pi\epsilon_0^2$, and $\ln\Lambda^{a/b}$ is the Coulomb logarithm. Using the relation

$$\nabla\nabla s = \frac{\boldsymbol{I}}{s} - \frac{\boldsymbol{s}\boldsymbol{s}}{s^3}, \tag{3}$$

Eq. (2) is written as

$$\boldsymbol{S}_c^{a/b} = \frac{c_{ab}}{m_a}\int d\boldsymbol{v}'\nabla\nabla s\cdot\left[\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') - \frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v})\right]. \tag{4}$$

The first term in Eq. (4) can be written as

$$\begin{aligned}\int d\boldsymbol{v}'\nabla\nabla s\cdot\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') &= \frac{f_a(\boldsymbol{v})}{m_b}\int d\boldsymbol{v}'\nabla\nabla s\cdot\nabla' f_b(\boldsymbol{v}') \\ &= -8\pi\frac{f_a(\boldsymbol{v})}{m_b}\nabla g(\boldsymbol{v}),\end{aligned} \tag{5}$$

where

$$g(\boldsymbol{v}) = -\frac{1}{4\pi}\int d\boldsymbol{v}'\frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}. \tag{6}$$

(Refer to Sec. (7) for the proof of Eq. (5).) The second term in Eq. (4) is written as

$$\begin{aligned}\int d\boldsymbol{v}'\nabla\nabla s\cdot\frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v}) &= \frac{1}{m_a}\left(\int d\boldsymbol{v}' f_b(\boldsymbol{v}')\nabla\nabla s\right)\cdot\nabla f_a(\boldsymbol{v}) \\ &= \frac{1}{m_a}\left(\nabla\nabla\int f_b(\boldsymbol{v}') s\, d\boldsymbol{v}'\right)\cdot\nabla f_a(\boldsymbol{v}) \\ &= -\frac{8\pi}{m_a}(\nabla\nabla h(\boldsymbol{v}))\cdot\nabla f_a(\boldsymbol{v}),\end{aligned} \tag{7}$$

where

$$h(\boldsymbol{v}) = -\frac{1}{8\pi}\int d\boldsymbol{v}' f_b(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|. \tag{8}$$

The functions $g_b(\boldsymbol{v})$ and $h_b(\boldsymbol{v})$ are called Rosenbluth potentials. (The reason that $g(\boldsymbol{v})$ is called "potential" is that $g_b(\boldsymbol{v})$ satisfies the following equation

$$\nabla^2 g(\boldsymbol{v}) = f_b(\boldsymbol{v}), \tag{9}$$

which indicates $g_b(\boldsymbol{v})$ is the "potential" produced by the "charge distribution $f_b(\boldsymbol{v})$".)

Using the above results, Eq. (4) is written as

$$\begin{aligned}\boldsymbol{S}_c^{a/b} &= -8\pi\frac{c_{ab}}{m_a}\left[\frac{1}{m_b}\nabla g(\boldsymbol{v}) f_a(\boldsymbol{v}) - \frac{1}{m_a}(\nabla\nabla h(\boldsymbol{v}))\cdot\nabla f_a(\boldsymbol{v})\right] \\ &= 8\pi\frac{c_{ab}}{m_a^2}\nabla\nabla h(\boldsymbol{v})\cdot\nabla f_a(\boldsymbol{v}) - 8\pi\frac{c_{ab}}{m_a m_b}\nabla g(\boldsymbol{v}) f_a(\boldsymbol{v}) \\ &= -\boldsymbol{D}_c^{a/b}\cdot\nabla f_a(\boldsymbol{v}) + \boldsymbol{F}_c^{a/b} f_a(\boldsymbol{v})\end{aligned} \tag{10}$$

where $\boldsymbol{D}_c^{a/b}$ and $\boldsymbol{F}_c^{a/b}$ are the diffusion tensor and friction vector, which are given respectively by

$$\begin{aligned}\boldsymbol{D}_c^{a/b} &= -8\pi\frac{c_{ab}}{m_a^2}\nabla\nabla h(\boldsymbol{v}) \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\nabla\nabla h(\boldsymbol{v})\end{aligned} \tag{11}$$

$$\begin{aligned}\boldsymbol{F}_c^{a/b} &= -8\pi\frac{c_{ab}}{m_a m_b}\nabla g(\boldsymbol{v}) \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\nabla g(\boldsymbol{v}),\end{aligned} \tag{12}$$

where

$$\Gamma^{a/b} = \frac{2n_b c_{ab}}{m_a^2} = \frac{n_b q_a^2 q_b^2}{4\pi\epsilon_0^2 m_a^2} \ln\Lambda^{a/b} \tag{13}$$

The form of collision flux in Eq. (10) is called the Fokker-Planck form.

Assuming axial symmetry for f_a and f_b, and using spherical coordinates $(v,\, \theta,\, \phi)$, then $f_a(\boldsymbol{v}) = f_a(v,\, \theta)$ and $f_b(\boldsymbol{v}) = f_b(v, \theta)$. The components of collision flux in Eq. (10) is written as

$$S_{cv}^{a/b} = -D_{cvv}^{a/b} \frac{\partial f_a}{\partial v} - D_{cv\theta}^{a/b} \frac{1}{v} \frac{\partial f_a}{\partial \theta} + F_{cv}^{a/b} f_a \tag{14}$$

$$S_{c\theta}^{a/b} = -D_{c\theta v}^{a/b} \frac{\partial f_a}{\partial v} - D_{c\theta\theta}^{a/b} \frac{1}{v} \frac{\partial f_a}{\partial \theta} + F_{c\theta}^{a/b} f_a \tag{15}$$

$$S_{c\phi}^{a/b} = 0. \tag{16}$$

1.1 Fokker-Planck coefficients

The components of the friction coefficient $\boldsymbol{F}_c^{a/b}$ in Eq. (12) are written as

$$\boldsymbol{F}_{cv}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \frac{m_a}{m_b} \frac{\partial g}{\partial v}, \tag{17}$$

$$\boldsymbol{F}_{c\theta}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \frac{m_a}{m_b} \frac{1}{v} \frac{\partial g}{\partial \theta}. \tag{18}$$

Next we calculate the components of the diffusion coefficient $\boldsymbol{D}_c^{a/b}$ in Eq. (11). Using

$$(\nabla\nabla h_b)_{vv} = \frac{\partial^2 h}{\partial v^2} \tag{19}$$

$$(\nabla\nabla h_b)_{v\theta} = (\nabla\nabla h_b)_{\theta v} = \frac{1}{v} \frac{\partial^2 h}{\partial\theta\partial v} - \frac{1}{v^2} \frac{\partial h}{\partial \theta} \tag{20}$$

$$(\nabla\nabla h_b)_{\theta\theta} = \frac{1}{v} \frac{\partial h}{\partial v} + \frac{1}{v^2} \frac{\partial^2 h}{\partial \theta^2} \tag{21}$$

$$(\nabla\nabla h_b)_{\phi\phi} = \frac{1}{v} \frac{\partial h}{\partial v} + \frac{\cos\theta}{v \sin\theta} \frac{1}{v} \frac{\partial h}{\partial \theta}, \tag{22}$$

(the derivation of these formulas are given in Sec. (4)) the components of the diffusion coefficient tensor are written as

$$D_{cvv}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \frac{\partial^2 h}{\partial v^2} \tag{23}$$

$$D_{cv\theta}^{a/b} = D_{c\theta v}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \left(\frac{1}{v} \frac{\partial^2 h}{\partial\theta\partial v} - \frac{1}{v^2} \frac{\partial h}{\partial \theta} \right) \tag{24}$$

$$D_{c\theta\theta}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \left(\frac{1}{v} \frac{\partial h}{\partial v} + \frac{1}{v^2} \frac{\partial^2 h}{\partial \theta^2} \right) \tag{25}$$

$$D_{c\phi\phi}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b} \left(\frac{1}{v} \frac{\partial h}{\partial v} + \frac{\cos\theta}{v \sin\theta} \frac{1}{v} \frac{\partial h}{\partial \theta} \right). \tag{26}$$

Although $D_{c\phi\phi}^{a/b}$ is not zero, it does not appear in the expression of collision flux since $D_{c\phi\phi}^{a/b}$ is multiplied by the ϕ component of ∇f_a, which is zero for axially symmetric f_a.

2 Legendre harmonics expansion

In the case of axial symmetric f_b, it can be proved that $g(\boldsymbol{v})$ and $h(\boldsymbol{v})$ are also axially symmetric, i.e., $g(\boldsymbol{v}) = g(v, \theta)$ and $h(\boldsymbol{v}) = h(v, \theta)$. Expand $f_b(v, \theta)$, $h(v, \theta)$, and $g(v, \theta)$ in terms of Legendre harmonics

$$f_b(v, \theta) = \sum_{l=0}^{\infty} f_b^{(l)}(v) P_l(\cos\theta), \tag{27}$$

$$g(v, \theta) = \sum_{l=0}^{\infty} g^{(l)}(v) P_l(\cos\theta), \tag{28}$$

$$h(v, \theta) = \sum_{l=0}^{\infty} h^{(l)}(v) P_l(\cos\theta), \tag{29}$$

where

$$f_b^{(l)}(v) = \frac{2l+1}{2}\int_0^\pi f_b(v,\theta)P_l(\cos\theta)\sin\theta d\theta, \tag{30}$$

$$g^{(l)}(v) = \frac{2l+1}{2}\int_0^\pi g(v,\theta)P_l(\cos\theta)\sin\theta d\theta. \tag{31}$$

$$h^{(l)}(v) = \frac{2l+1}{2}\int_0^\pi h(v,\theta)P_l(\cos\theta)\sin\theta d\theta. \tag{32}$$

then the expansion coefficient of $g^l(v)$ and $h^l(v)$ have the following relation with $f_b^l(v)$ (the proof is given in another note).

$$g^{(l)}(v) = -\frac{1}{2l+1}\left[\int_0^v \frac{(v')^{l+2}}{v^{l+1}} f_b^{(l)}(v')dv' + \int_v^\infty \frac{v^l}{(v')^{l-1}} f_b^{(l)}(v')dv'\right] \tag{33}$$

$$\begin{aligned} h^{(l)}(v) &= \frac{1}{2(4l^2-1)}\left[\int_0^v \frac{(v')^{l+2}}{v^{l-1}}\left(1-\frac{2l-1}{2l+3}\frac{v'^2}{v^2}\right) f_b^{(l)}(v')dv' \right.\\ &+ \left.\int_v^\infty \frac{v^l}{(v')^{l-3}}\left(1-\frac{2l-1}{2l+3}\frac{v^2}{v'^2}\right) f_b^{(l)}(v')dv'\right]. \end{aligned} \tag{34}$$

(In the numerical calculation of the collision flux in Eq. (10) on two-dimension plane (v,θ) with $N\times N$ grids, it is easy to find that by using the Legendre harmonics expansion, the calculation of collision flux only takes $K\times N^3$ operations, where $K<N$, with K the number of Legendre harmonics used in the expansion)

Using the above results, the friction and diffusion coefficients are written respectively as

$$\begin{aligned} F_{cv}^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{\partial g}{\partial v} \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\sum_{l=0}^{\infty}\frac{\partial g^{(l)}(v)}{\partial v}P_l(\cos\theta) \end{aligned} \tag{35}$$

$$\begin{aligned} F_{c\theta}^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{1}{v}\frac{\partial g}{\partial \theta} \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{1}{v}\sum_{l=0}^{\infty} g^{(l)}(v)\frac{\partial P_l(\cos\theta)}{\partial\theta} \end{aligned} \tag{36}$$

$$\begin{aligned} D_{cvv}^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{\partial^2 h}{\partial v^2} \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\sum_{l=0}^{\infty}\frac{\partial^2 h^{(l)}}{\partial v^2}P_l(\cos\theta) \end{aligned} \tag{37}$$

$$\begin{aligned} D_{cv\theta}^{a/b} = D_{c\theta v}^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\left(\frac{1}{v}\frac{\partial^2 h}{\partial\theta\partial v} - \frac{1}{v^2}\frac{\partial h}{\partial\theta}\right) \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\sum_{l=0}^{\infty}\left[\frac{1}{v}\frac{\partial h^{(l)}(v)}{\partial v} - \frac{1}{v^2}h^{(l)}(v)\right]\frac{\partial P_l(\cos\theta)}{\partial\theta} \end{aligned} \tag{38}$$

$$\begin{aligned} D_{c\theta\theta}^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\left(\frac{1}{v}\frac{\partial h}{\partial v} + \frac{1}{v^2}\frac{\partial^2 h}{\partial\theta^2}\right) \\ &= -\frac{4\pi\Gamma^{a/b}}{n_b}\sum_{l=0}^{\infty}\left(\frac{1}{v}\frac{\partial h^{(l)}}{\partial v}P_l(\cos\theta) + \frac{1}{v^2}h^{(l)}(v)\frac{\partial^2 P_l(\cos\theta)}{\partial\theta^2}\right) \end{aligned} \tag{39}$$

2.1 Collision term

The components of the collision flux are written as

$$S_{cv}^{a/b} = -D_{cvv}^{a/b}\frac{\partial f_a}{\partial u} - D_{cv\theta}^{a/b}\frac{1}{u}\frac{\partial f_a}{\partial\theta} + F_{cv}^{a/b}f_a, \tag{40}$$

$$S_{c\theta}^{a/b} = -D_{c\theta v}^{a/b}\frac{\partial f_a}{\partial u} - D_{c\theta\theta}^{a/b}\frac{1}{u}\frac{\partial f_a}{\partial\theta} + F_{c\theta}^{a/b}f_a, \tag{41}$$

The collision term is the divergence of the collsion flux

$$\begin{aligned} -C^{a/b}(f_a,f_b) &= \nabla_u\cdot\boldsymbol{S}_c \\ &= \frac{1}{u^2}\frac{\partial}{\partial u}u^2 S_{cv} + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_{c\theta}. \end{aligned} \tag{42}$$

For isotropic distribution f_a, the collision flux is simplified to

$$S_{cv}^{a/b} = -D_{cvv}^{a/b}\frac{\partial f_a}{\partial u} + F_{cv}^{a/b} f_a, \tag{43}$$

and

$$S_{c\theta}^{a/b} = -D_{c\theta v}^{a/b}\frac{\partial f_a}{\partial u} + F_{c\theta}^{a/b} f_a. \tag{44}$$

Thus the collision term is written as

$$-C^{a/b}(f_a, f_b) = \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(-D_{cvv}^{a/b}\frac{\partial f_a}{\partial u} + F_{cv}^{a/b} f_a\right) + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(-D_{c\theta v}^{a/b}\frac{\partial f_a}{\partial u} + F_{c\theta}^{a/b} f_a\right). \tag{45}$$

Then, for the lth Legendre harmonic $f_b^{(l)} P_l(\cos\theta)$, the collision term is written as

$$\begin{aligned}-C^{a/b}(f_a, f_b^{(l)} P_l(\cos\theta)) \;=\; & \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left[\frac{4\pi\Gamma^{a/b}}{n_b}\left(\frac{\partial^2 h^{(l)}}{\partial v^2}P_l(\cos\theta)\right)\frac{\partial f_a}{\partial u} - \frac{4\pi\Gamma^{a/b}}{n_b}\left(\frac{m_a}{m_b}\frac{\partial g^{(l)}(v)}{\partial v}P_l(\cos\theta)\right) f_a\right] \\ & + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left\{\frac{4\pi\Gamma^{a/b}}{n_b}\left[\frac{1}{v}\frac{\partial h^{(l)}(v)}{\partial v} - \frac{1}{v^2}h^{(l)}(v)\right]\frac{\partial P_l(\cos\theta)}{\partial\theta}\frac{\partial f_a}{\partial u} - \right. \\ & \left. \frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{1}{v}\, g^{(l)}(v)\frac{\partial P_l(\cos\theta)}{\partial\theta} f_a\right\} \end{aligned} \tag{46}$$

Then

$$\begin{aligned}-\frac{C^{a/b}(f_a, f_b^{(l)} P_l(\cos\theta))}{P_l(\cos\theta)} \;=\; & \frac{4\pi\Gamma^{a/b}}{n_b}\frac{1}{u^2}\frac{\partial}{\partial u}u^2\left[\frac{\partial^2 h^{(l)}}{\partial v^2}\frac{\partial f_a}{\partial u} - \frac{m_a}{m_b}\frac{\partial g^{(l)}(v)}{\partial v} f_a\right] + \frac{4\pi\Gamma^{a/b}}{n_b}\frac{1}{u}\left\{\left[\frac{1}{v}\frac{\partial h^{(l)}(v)}{\partial v} - \frac{1}{v^2}h^{(l)}(v)\right]\frac{\partial f_a}{\partial u} - \right. \\ & \left. \frac{m_a}{m_b}\frac{1}{v}\, g^{(l)}(v) f_a\right\}[-l(l+1)], \end{aligned} \tag{47}$$

which is independent of the pitch angle θ.

$$\left(-\frac{v}{v_{te}^2}\right)D_{uu} = F_v$$

2.2 Isotropic background distribution

If the background distribution is isotropic $f_b(\boldsymbol{v}) = f_b(v)$, then so are g_b and h_b (Proof omitted). Then the collision flux is given by

$$\boldsymbol{S}_c^{a/b} = -\boldsymbol{D}_c^{a/b}\cdot\nabla f_a(\boldsymbol{v}) + \boldsymbol{F}_c^{a/b} f_a(\boldsymbol{v}) \tag{48}$$

with

$$D_{cvv}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{\partial^2 h}{\partial v^2}, \tag{49}$$

$$D_{c\theta\theta}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{1}{v}\frac{\partial h}{\partial v}, \tag{50}$$

$$D_{c\phi\phi}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{1}{v}\frac{\partial h}{\partial v}, \tag{51}$$

$$F_{cv}^{a/b} = -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{\partial g}{\partial v}, \tag{52}$$

and all the other components are zeroes. In this case, $h(v)$ and $g(v)$ are related with $f_b(v)$ by

$$\begin{aligned} h(v) \;=\; & -\frac{1}{2}\left[\int_0^v (v')^2 v\left(1 + \frac{(v')^2/2}{3v^2/2}\right) f_b(v')dv' \right. \\ & \left. + \int_v^\infty (v')^3\left(1 + \frac{v^2/2}{3(v')^2/2}\right) f_b(v')dv'\right]. \end{aligned} \tag{53}$$

$$g(v) = -\left[\int_0^v \frac{(v')^2}{v} f_b(v')dv' + \int_v^\infty v' f_b(v')dv'\right] \tag{54}$$

From this we can calculate the derivative of $h(v)$ and $g(v)$, (using Wolfram Mathematic),

$$\frac{\partial h(v)}{\partial v} = -\left[\int_0^v \frac{v'^2}{6v^2}\big(3v^2-(v')^2\big) f_b(v')dv' + \int_v^\infty \frac{v}{3} v' f_b(v')dv'\right], \tag{55}$$

$$\frac{\partial^2 h(v)}{\partial v^2} = -\int_0^v \frac{(v')^4}{3v^3} f_b(v')dv' - \int_v^\infty \frac{1}{3} v' f_b(v')dv', \tag{56}$$

$$\frac{\partial g}{\partial v} = -\int_0^v \frac{(v')^2}{v^2} f_b(v')dv'. \tag{57}$$

Therefore

$$D_{cvv}^{a/b}(v) = \frac{4\pi\Gamma^{a/b}}{3n_b}\left(\int_0^v \frac{(v')^4}{v^3} f_b(v')dv' + \int_v^\infty v' f_b(v')dv'\right) \tag{58}$$

$$D_{c\theta\theta}^{a/b}(v) = \frac{4\pi\Gamma^{a/b}}{3n_b}\left[\int_0^v \frac{v'^2}{2v^3}\big(3v^2-(v')^2\big) f_b(v')dv' + \int_v^\infty v' f_b(v')dv'\right] \tag{59}$$

$$F_{cv}^{a/b}(v) = -\frac{4\pi\Gamma^{a/b}}{3n_b}\frac{m_a}{m_b}\int_0^v \frac{3(v')^2}{v^2} f_b(v')dv' \tag{60}$$

For axial symmetric $f_a = f_a(v,\theta)$, collision flux is given by

$$S_{cv}^{a/b} = -D_{cvv}^{a/b}\frac{\partial f_a(v,\theta)}{\partial v} + F_{cv}^{a/b} f_a(v,\theta) \tag{61}$$

$$S_{c\theta}^{a/b} = -D_{c\theta\theta}^{a/b}\frac{1}{v}\frac{\partial f_a(v,\theta)}{\partial \theta} \tag{62}$$

$$S_{c\phi}^{a/b} = 0 \tag{63}$$

The corresponding collision term is written

$$\begin{aligned} C(f_a, f_b) &= -\nabla\cdot \boldsymbol{S}_c^{a/b} \\ &= -\left[\frac{1}{v^2}\frac{\partial}{\partial v} v^2 S_v^{a/b} + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_\theta^{a/b}\right] \\ &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2\left(D_{cvv}^{a/b}\frac{\partial f_a}{\partial v} - F_{cv}^{a/b} f_a\right) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(D_{c\theta\theta}^{a/b}\frac{1}{v}\frac{\partial f_a}{\partial\theta}\right)\right] \\ &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2\left(D_{cvv}^{a/b}\frac{\partial f_a}{\partial v} - F_{cv}^{a/b} f_a\right) + \frac{D_{c\theta\theta}^{a/b}}{v^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial f_a}{\partial\mu}\right], \end{aligned} \tag{64}$$

where $\mu = \cos\theta$.

2.3 One-dimension problem

If f_a is also isotropic, i.e., $f_a(\boldsymbol{v}) = f_b(v)$, collision flux in Eqs. (61)-(63) reduce to

$$\begin{aligned} S_{cv}^{a/b} &= -D_{cvv}^{a/b}\frac{\partial f_a(v)}{\partial v} + F_{cv}^{a/b} f_a(v), \\ S_{c\theta}^{a/b} &= 0, \\ S_{c\phi}^{a/b} &= 0, \end{aligned}$$

where $D_{cvv}^{a/b}$ and $F_{cv}^{a/b}$ are given by Eqs. (58) and (60), respectively. For isotropic $f_i(v)$ and $f_e(v)$, electron-ion collision flux $\boldsymbol{S}_c^{e/i}$can be neglected (Proof is given in another note). Therefore

$$\frac{\partial f_e}{\partial t} = -\nabla_{\boldsymbol{v}}\cdot \boldsymbol{S}_c^{e/e} = -\nabla_{\boldsymbol{v}}\cdot\left(S_{cv}^{e/e}\boldsymbol{e}_v\right) \tag{65}$$

2.4 The high-velocity limit

When $v \gg v_{tb}$, Eqs. (58), (59), and (60) can be approximated respectively by

$$D_{cvv}^{a/b}(v) = \frac{4\pi\Gamma^{a/b}}{3n_b}\frac{1}{v^3}\int_0^\infty (v')^4 f_b(v')dv' \tag{66}$$

$$D_{c\theta\theta}^{a/b}(v) == \frac{4\pi\Gamma^{a/b}}{3n_b}\frac{1}{2v^3}\int_0^\infty (v')^2\big(3v^2-(v')^2\big) f_b(v')dv' \tag{67}$$

$$F_{cv}^{a/b}(v) = -\frac{4\pi\Gamma^{a/b}}{3n_b}\frac{m_a}{m_b}\frac{1}{v^2}\int_0^\infty 3(v')^2 f_b(v')dv' \tag{68}$$

Using

$$n_b = \int f_b(v)d^3\boldsymbol{v} = 4\pi\int f_b(v)v^2dv \tag{69}$$

and

$$v_{tb}^2 = \frac{4\pi}{3n_b}\int_0^\infty v^4 f_b(v)dv. \tag{70}$$

one obtains

$$D_{cvv}^{a/b}(v) = \Gamma^{a/b}\frac{v_{tb}^2}{v^3} \tag{71}$$

$$D_{c\theta\theta}^{a/b}(v) = \Gamma^{a/b}\frac{1}{2v}\left(1 - \frac{v_{tb}^2}{v^2}\right) \tag{72}$$

$$F_{cv}^{a/b}(v) = -\Gamma^{a/b}\frac{m_a}{m_b}\frac{1}{v^2} \tag{73}$$

Eqs. (71), (72), and (73) are the high-velocity limit of the diffusion and friction coefficients. The corresponding collision operator is written as

$$\begin{aligned} C(f_a, f_m) &= -\nabla\cdot\boldsymbol{S}^{a/m} \\ &= \Gamma^{a/b}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v_{tb}^2}{v}\frac{\partial f_a}{\partial v} + f_a\right) + \frac{1}{2v^3}\left(1 - \frac{v_{tb}^2}{v^2}\right)\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f_a}{\partial\mu}\right] \end{aligned} \tag{74}$$

This is Eq.(2.13) in Fisch's 1987 review paper[1]. (We note in passing that, in the relativistic case, the corresponding collision operater takes the following form[2, 3]

$$C(f_a, f_m) = \frac{1}{u^2}\frac{\partial}{\partial u}u^2\lambda_s(u)f + [\nu_{ei}(u) + \nu_D(u)]\frac{1}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f_a}{\partial\mu} \tag{75}$$

Refer to Lin-Liu's paper[3] for the expression of the coefficients apprearing in above eqution.)

For electron-ion collision, using the assumption, $v_{ti}/v \to 0$, $m_e/m_i \to 0$, the diffusion and friction coefficients, Eqs. (71), (72), and (73) are further reduced to,

$$D_{cvv}^{a/b}(v) = 0 \tag{76}$$

$$D_{c\theta\theta}^{a/b}(v) = \Gamma^{a/b}\frac{1}{2v} \tag{77}$$

$$F_{cv}^{a/b}(v) = 0 \tag{78}$$

The corresponding collision operator is written as

$$\begin{aligned} C(f_e, f_i) &= \Gamma^{e/i}\left[\frac{1}{2v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f_e}{\partial\mu}\right] \\ &= Z_{\text{eff}}\Gamma^{e/e}\left[\frac{1}{2v^3}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f_e}{\partial\mu}\right] \end{aligned} \tag{79}$$

This is Eq.(37) in Karney's 1986 paper[4]. Here

$$Z_{\text{eff}} \equiv \frac{\Gamma^{e/i}}{\Gamma^{e/e}} = \frac{n_i q_i^2 \ln\Lambda^{e/i}}{n_e q_e^2 \ln\Lambda^{e/e}}.$$

From Eq. (79), one can easily prove that

$$\int v^2 C(f_e, f_i)d^3\boldsymbol{v} = 0, \tag{80}$$

which indicates this collision operator does not change the energy of electrons in electron-ion collision.

Define the Lorentz operator,

$$L \equiv \frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} \Leftrightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}$$

It can be proved that (refer to another note for the proof),

$$\int_0^\pi fL(g)\sin\theta d\theta = \int_0^\pi gL(f)\sin\theta d\theta$$

Using this, one can prove that operator in Eq. (79) is self adjoint,

$$\int gC^{e/i}(f)d^3\boldsymbol{v} = \int fC^{e/i}(g)d^3\boldsymbol{v} \tag{81}$$

and

$$\int g C^{e/i}(f f_m) d^3\boldsymbol{v} = \int f C^{e/i}(g f_m) d^3\boldsymbol{v} \tag{82}$$

3 Conservation laws

In this section, I will prove the conservation laws of the collision operator in the Landau form.

Conservation of number,

$$\begin{aligned}\int C(f_a, f_b) d^3\boldsymbol{v} &= -\int \nabla\cdot \boldsymbol{S}^{a/b} d^3\boldsymbol{v} \\ &= -\oint \boldsymbol{S}^{a/b} d^2\boldsymbol{\sigma} \\ &= 0\end{aligned}$$

Now consider the conservation of momentum,

$$\int m_a \boldsymbol{v}\, C(f_a, f_b) d^3\boldsymbol{v} + \int m_b \boldsymbol{v}\, C(f_b, f_a) d^3\boldsymbol{v} = -\int \boldsymbol{v}\Big[\nabla\cdot\Big(m_a \boldsymbol{S}^{a/b} + m_b \boldsymbol{S}^{b/a}\Big)\Big] d^3\boldsymbol{v} \tag{83}$$

Using the Landau form,

$$\boldsymbol{S}^{a/b} = \frac{c_{ab}}{m_a}\int d^3\boldsymbol{v}'\boldsymbol{U}\cdot\left[\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') - \frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v})\right],$$

and noting that c_{ab} and the collision kernel $\boldsymbol{U}$ does not change when the species a and b are exchanged, one knows that,

$$m_a \boldsymbol{S}^{a/b} = c_{ab}\int d^3\boldsymbol{v}'\boldsymbol{U}\cdot\left[\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') - \frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v})\right]$$

$$m_b \boldsymbol{S}^{b/a} = c_{ba}\int d^3\boldsymbol{v}'\boldsymbol{U}\cdot\left[\frac{f_b(\boldsymbol{v})}{m_a}\nabla' f_a(\boldsymbol{v}') - \frac{f_a(\boldsymbol{v}')}{m_b}\nabla f_b(\boldsymbol{v})\right]$$

Therefore,

$$\begin{aligned}m_a \boldsymbol{S}^{a/b} + m_b \boldsymbol{S}^{b/a} &= c_{ab}\int d^3\boldsymbol{v}'\boldsymbol{U}\cdot\left[\frac{f_a(\boldsymbol{v})}{m_b}\nabla' f_b(\boldsymbol{v}') - \frac{f_b(\boldsymbol{v}')}{m_a}\nabla f_a(\boldsymbol{v}) + \frac{f_b(\boldsymbol{v})}{m_a}\nabla' f_a(\boldsymbol{v}') - \frac{f_a(\boldsymbol{v}')}{m_b}\nabla f_b(\boldsymbol{v})\right] \\ &= \end{aligned}$$

Right side of Eq.(83) reduces to,

$$-\int \boldsymbol{v}\Big[\nabla\cdot\Big(m_a \boldsymbol{S}^{a/b} + m_b \boldsymbol{S}^{b/a}\Big)\Big] d^3\boldsymbol{v} =$$

4 Derivation of the components of the diffusion tensor

First we consider the gradient of a vector. In spherical coordinates (v, θ, ϕ), a vector can be written as

$$\boldsymbol{A} = A_v\hat{\boldsymbol{v}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}. \tag{84}$$

Then the gradient of $\boldsymbol{A}$ is written as

$$\begin{aligned}\nabla\boldsymbol{A} &= \nabla(A_v\hat{\boldsymbol{v}} + A_\theta\hat{\boldsymbol{\theta}} + A_\phi\hat{\boldsymbol{\phi}}) \\ &= (\nabla A_v)\hat{\boldsymbol{v}} + A_v\nabla\hat{\boldsymbol{v}} + (\nabla A_\theta)\hat{\boldsymbol{\theta}} + A_\theta\nabla\hat{\boldsymbol{\theta}} + (\nabla A_\phi)\hat{\boldsymbol{\phi}} + A_\phi\nabla\hat{\boldsymbol{\phi}}\end{aligned} \tag{85}$$

The unit vectors depend on both θ and ϕ, The non-zero derivatives are:

$$\begin{aligned}\frac{\partial\hat{\boldsymbol{v}}}{\partial\phi} &= \sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial\hat{\boldsymbol{v}}}{\partial\theta} &= \hat{\boldsymbol{\theta}} \\ \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} &= \cos\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} &= -\hat{\boldsymbol{v}} \\ \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} &= -\cos\theta\hat{\boldsymbol{\theta}} - \sin\theta\hat{\boldsymbol{v}}\end{aligned}$$

Using these results, we obtain

$$\begin{aligned}
\nabla \hat{\boldsymbol{v}} &= \left(\hat{\boldsymbol{v}} \frac{\partial}{\partial v} + \hat{\boldsymbol{\theta}} \frac{1}{v} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \frac{\partial}{\partial \phi} \right) \hat{\boldsymbol{v}} \\
&= \hat{\boldsymbol{v}} \frac{\partial \hat{\boldsymbol{v}}}{\partial v} + \hat{\boldsymbol{\theta}} \frac{1}{v} \frac{\partial \hat{\boldsymbol{v}}}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \frac{\partial \hat{\boldsymbol{v}}}{\partial \phi} \\
&= 0 + \hat{\boldsymbol{\theta}} \frac{1}{v} \hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \sin\theta \hat{\boldsymbol{\phi}} \\
&= \frac{1}{v} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\theta}} + \frac{1}{v} \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\nabla \hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{v}} \frac{\partial \hat{\boldsymbol{\theta}}}{\partial v} + \hat{\boldsymbol{\theta}} \frac{1}{v} \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \frac{\partial \hat{\boldsymbol{\theta}}}{\partial \phi} \\
&= 0 + \hat{\boldsymbol{\theta}} \frac{1}{v} (-\hat{\boldsymbol{v}}) + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} (\cos\theta \hat{\boldsymbol{\phi}}) \\
&= -\frac{1}{v} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{v}} + \frac{\cos\theta}{v \sin\theta} \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\phi}}
\end{aligned}$$

$$\begin{aligned}
\nabla \hat{\boldsymbol{\phi}} &= \hat{\boldsymbol{v}} \frac{\partial \hat{\boldsymbol{\phi}}}{\partial v} + \hat{\boldsymbol{\theta}} \frac{1}{v} \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} \\
&= 0 + 0 + \hat{\boldsymbol{\phi}} \frac{1}{v \sin\theta} \left(-\cos\theta \hat{\boldsymbol{\theta}} - \sin\theta \hat{\boldsymbol{v}} \right) \\
&= -\frac{\cos\theta}{v \sin\theta} \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\theta}} - \frac{1}{v} \hat{\boldsymbol{\phi}} \hat{\boldsymbol{v}}
\end{aligned}$$

Assuming axial symmetry, we have

$$\begin{aligned}
\nabla A_v &= \frac{\partial A_v}{\partial v} \hat{\boldsymbol{v}} + \frac{1}{v} \frac{\partial A_v}{\partial \theta} \hat{\boldsymbol{\theta}} \\
\nabla A_\theta &= \frac{\partial A_\theta}{\partial v} \hat{\boldsymbol{v}} + \frac{1}{v} \frac{\partial A_\theta}{\partial \theta} \hat{\boldsymbol{\theta}} \\
\nabla A_\phi &= \frac{\partial A_\phi}{\partial v} \hat{\boldsymbol{v}} + \frac{1}{v} \frac{\partial A_\phi}{\partial \theta} \hat{\boldsymbol{\theta}}
\end{aligned}$$

Using the above result, Eq. (85) reduce to

$$\begin{aligned}
\nabla \boldsymbol{A} &= (\nabla A_v) \hat{\boldsymbol{v}} + A_v \nabla \hat{\boldsymbol{v}} + (\nabla A_\theta) \hat{\boldsymbol{\theta}} + A_\theta \nabla \hat{\boldsymbol{\theta}} + (\nabla A_\phi) \hat{\boldsymbol{\phi}} + A_\phi \nabla \hat{\boldsymbol{\phi}} \\
&= \frac{\partial A_v}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{v}} + \frac{1}{v} \frac{\partial A_v}{\partial \theta} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{v}} + A_v \frac{1}{v} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\theta}} + A_v \frac{1}{v} \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\phi}} + \frac{\partial A_\theta}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{\theta}} + \frac{1}{v} \frac{\partial A_\theta}{\partial \theta} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\theta}} \\
&+ -\frac{1}{v} A_\theta \hat{\boldsymbol{\theta}} \hat{\boldsymbol{v}} + \frac{\cos\theta}{v \sin\theta} A_\theta \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\phi}} + \frac{\partial A_\phi}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{\phi}} + \frac{1}{v} \frac{\partial A_\phi}{\partial \theta} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\phi}} \\
&- \frac{\cos\theta}{v \sin\theta} A_\phi \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\theta}} - \frac{1}{v} A_\phi \hat{\boldsymbol{\phi}} \hat{\boldsymbol{v}} \\
&= \frac{\partial A_v}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{v}} + \frac{\partial A_\theta}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{\theta}} + \frac{\partial A_\phi}{\partial v} \hat{\boldsymbol{v}} \hat{\boldsymbol{\phi}} \\
&+ \left(\frac{1}{v} \frac{\partial A_v}{\partial \theta} - \frac{1}{v} A_\theta \right) \hat{\boldsymbol{\theta}} \hat{\boldsymbol{v}} + \left(A_v \frac{1}{v} + \frac{1}{v} \frac{\partial A_\theta}{\partial \theta} \right) \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\theta}} + \frac{1}{v} \frac{\partial A_\phi}{\partial \theta} \hat{\boldsymbol{\theta}} \hat{\boldsymbol{\phi}} \\
&- \frac{1}{v} A_\phi \hat{\boldsymbol{\phi}} \hat{\boldsymbol{v}} - \frac{\cos\theta}{v \sin\theta} A_\phi \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\theta}} + \left(A_v \frac{1}{v} + \frac{\cos\theta}{v \sin\theta} A_\theta \right) \hat{\boldsymbol{\phi}} \hat{\boldsymbol{\phi}}
\end{aligned} \tag{86}$$

Now supposing $\boldsymbol{A}$ is the gradient of a scalar h_b that is independent of ϕ, we have

$$\begin{aligned}
A_v &= \frac{\partial h}{\partial v} \\
A_\theta &= \frac{1}{v} \frac{\partial h}{\partial \theta} \\
A_\phi &= 0
\end{aligned}$$

Substitute these into Eq. (86), we obtain

$$(\nabla \nabla h_b)_{vv} = \frac{\partial^2 h}{\partial v^2} \tag{87}$$

$$(\nabla \nabla h_b)_{v\theta} = (\nabla \nabla h_b)_{\theta v} = \frac{1}{v} \frac{\partial^2 h}{\partial \theta \partial v} - \frac{1}{v^2} \frac{\partial h}{\partial \theta} \tag{88}$$

$$(\nabla \nabla h_b)_{\theta\theta} = \frac{1}{v} \frac{\partial h}{\partial v} + \frac{1}{v^2} \frac{\partial^2 h}{\partial \theta^2} \tag{89}$$

$$(\nabla \nabla h_b)_{\phi\phi} = \frac{1}{v} \frac{\partial h}{\partial v} + \frac{\cos\theta}{v \sin\theta} \frac{1}{v} \frac{\partial h}{\partial \theta} \tag{90}$$

The other components of $\nabla \nabla h$ are zeroes. Note that $\nabla \nabla h$ still has non-zero $\phi\phi$ component even though h is independent of ϕ. Therefore the diffusion coefficients are written as

5 Landau form of relativistic collision operator

Relativistic collision operator takes the following form,

$$C(f_a, f_b) = -\nabla_u \cdot \boldsymbol{S}$$

$$\boldsymbol{S} = -\frac{c_{ab}}{m_a}\int \boldsymbol{U}\cdot\left(\frac{f_b(\boldsymbol{u}')}{m_a}\frac{\partial f_a(\boldsymbol{u})}{\partial \boldsymbol{u}} - \frac{f_a(\boldsymbol{u})}{m_b}\frac{\partial f_b(\boldsymbol{u}')}{\partial \boldsymbol{u}'}\right)d^3\boldsymbol{u}' \tag{91}$$

$$c_{ab} = \frac{q_a^2 q_b^2}{8\pi\epsilon_0^2}\ln\Lambda^{a/b}. \tag{92}$$

Eq.(91) can be further written as

$$\boldsymbol{S} = -\frac{\Gamma^{a/b}}{2n_b}\int \boldsymbol{U}\cdot\left(f_b(\boldsymbol{u}')\frac{\partial f_a(\boldsymbol{u})}{\partial \boldsymbol{u}} - \frac{m_a}{m_b}f_a(\boldsymbol{u})\frac{\partial f_b(\boldsymbol{u}')}{\partial \boldsymbol{u}'}\right)d^3\boldsymbol{u}'$$

where

$$\Gamma^{a/b} = \frac{n_b q_a^2 q_b^2}{4\pi\epsilon_0^2 m_a^2}\ln\Lambda^{a/b} \tag{93}$$

6 Fokker-Planck form of Relativistic collision operator

The gerneral form of Fokker-Planck collision term is given by,

$$C(f_a, f_b) = -\frac{\partial}{\partial \boldsymbol{u}}\cdot\left[-\boldsymbol{D}^{a/b}\cdot\frac{\partial f_a}{\partial \boldsymbol{u}} + \boldsymbol{F}^{a/b}f_a\right], \tag{94}$$

where $\boldsymbol{D}^{a/b}$ and $\boldsymbol{F}^{a/b}$ are given by

$$\boldsymbol{D}^{a/b}(\boldsymbol{u}) = \frac{c_{ab}}{m_a^2}\int \boldsymbol{U}(\boldsymbol{u},\boldsymbol{u}')f_b(\boldsymbol{u}')d\boldsymbol{u}', \tag{95}$$

$$\boldsymbol{F}^{a/b}(\boldsymbol{u}) = -\frac{c_{ab}}{m_a m_b}\int \frac{\partial}{\partial \boldsymbol{u}'}\cdot\boldsymbol{U}(\boldsymbol{u},\boldsymbol{u}')f_b(\boldsymbol{u}')d\boldsymbol{u}', \tag{96}$$

where $\boldsymbol{u}$ is the ratio of momentum to species mass,

$$c_{ab} = \frac{q_a^2 q_b^2}{8\pi\epsilon_0^2}\ln\Lambda^{a/b}.$$

The kernel $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \frac{\partial^2 s}{\partial \boldsymbol{v}\partial \boldsymbol{v}} \tag{97}$$

where $\boldsymbol{s} = \boldsymbol{v} - \boldsymbol{v}'$. The Rosenbluth potentials are given by

$$h_b(\boldsymbol{v}) = -\frac{1}{8\pi}\int d\boldsymbol{v}' s\gamma'^5 f_b(\boldsymbol{v}'), \tag{98}$$

$$g_b(\boldsymbol{v}) = \frac{1}{4\pi}\int d\boldsymbol{v}'\left[-\frac{1}{s}\left(\frac{1}{\gamma'} + \frac{1}{\gamma'^3}\right) - \frac{v'^2(v\cos\alpha - v')^2}{s^3\gamma'}\right]\gamma'^5 f_b(\boldsymbol{v}') \tag{99}$$

then

$$\boldsymbol{D}^{a/b}(\boldsymbol{u}) = -\frac{8\pi c_{ab}}{m_a^2}\frac{\partial^2 h_b(\boldsymbol{v})}{\partial \boldsymbol{v}\partial \boldsymbol{v}}. \tag{100}$$

$$\boldsymbol{F}^{a/b}(\boldsymbol{u}) = -\frac{4\pi c_{ab}}{m_a m_b}\frac{\partial}{\partial \boldsymbol{v}}g_b(\boldsymbol{v}). \tag{101}$$

$$S_v = \left[-\boldsymbol{D}^{a/b}\cdot\frac{\partial f_a}{\partial \boldsymbol{u}} + \frac{\partial g_b}{\partial v}f_a\right]$$

7 Proof of Eq. (5)

Eq. (5) is

$$\int d\boldsymbol{v}'\nabla\nabla s\cdot\nabla' f_b(\boldsymbol{v}') = -8\pi g(\boldsymbol{v}). \tag{102}$$

Proof: Define $\boldsymbol{a} = \nabla u$ and $\boldsymbol{b} = \nabla' f_b(\boldsymbol{v}')$, then

$$\begin{aligned}\nabla(\boldsymbol{a}\cdot b) &= (\nabla\boldsymbol{a})\cdot\boldsymbol{b} + (\nabla\boldsymbol{b})\cdot\boldsymbol{a}\\ &= (\nabla\boldsymbol{a})\cdot\boldsymbol{b}\end{aligned}$$

$$\begin{aligned}\int d\boldsymbol{v}'\nabla\nabla u\cdot\nabla' f_b(\boldsymbol{v}') &= \int \nabla\boldsymbol{a}\cdot\boldsymbol{b}d\boldsymbol{v}'\\ &= \int \nabla(\boldsymbol{a}\cdot\boldsymbol{b})d\boldsymbol{v}'\\ &= \nabla\int (\boldsymbol{a}\cdot\boldsymbol{b})d\boldsymbol{v}'\\ &= \nabla\int (\nabla s\cdot\nabla' f_b(\boldsymbol{v}'))d\boldsymbol{v}'\\ &= \nabla\int \left(\frac{\boldsymbol{s}}{s}\cdot\nabla' f_b(\boldsymbol{v}')\right)d\boldsymbol{v}' \qquad (103)\end{aligned}$$

The integration in the above equation can be further written as

$$\begin{aligned}\int \frac{\boldsymbol{s}}{s}\cdot\nabla' f_b(\boldsymbol{v}')d\boldsymbol{v}' &= \int \nabla'\cdot\left(\frac{\boldsymbol{s}}{s}f_b(\boldsymbol{v}')\right)d\boldsymbol{v}' - \int \nabla'\cdot\left(\frac{\boldsymbol{s}}{s}\right)f_b(\boldsymbol{v}')d\boldsymbol{v}'\\ &= 0-\int \nabla'\cdot\left(\frac{\boldsymbol{s}}{s}\right)f_b(\boldsymbol{v}')d\boldsymbol{v}'\\ &= \int \frac{2}{s}f_b(\boldsymbol{v}')d\boldsymbol{v}'\\ &= -8\pi g(\boldsymbol{v}),\end{aligned}$$

where use has been made of

$$\nabla'\cdot\left(\frac{\boldsymbol{s}}{s}\right)=\nabla'\left(\frac{1}{s}\right)\cdot\boldsymbol{s}+(\nabla'\cdot\boldsymbol{s})\frac{1}{s}=\frac{\boldsymbol{s}}{s^3}\cdot\boldsymbol{s}-3\frac{1}{s}=-\frac{2}{s}.$$

end of proof.

8 Calculation of $C(f_{am}, f^1(v)\cos\theta)$

$$\begin{aligned}\frac{h_b(v,\theta)}{\cos\theta} &= h_b^1(v)\\ &= \frac{1}{6}\left[\int_0^v (v')^3\left(1-\frac{(v')^2}{5v^2}\right)f_b^1(v')dv'\right.\\ &+ \left.\int_v^\infty v(v')^2\left(1-\frac{v^2}{5(v')^2}\right)f_b^1(v')dv'\right]\\ &= \frac{1}{6}\left[\int_0^v (v')^3 f_b^1(v')dv' - \frac{1}{5v^2}\int_0^v (v')^5 f_b^1(v')dv'\right.\\ &+ \left. v\int_v^\infty (v')^2 f_b^1(v')dv' - \frac{v^3}{5}\int_v^\infty f_b^1(v')dv'\right]\end{aligned}$$

$$\begin{aligned}6\frac{\partial h_b^1(v)}{\partial v} &= v^3 f_b^1(v)\\ &- \frac{1}{5v^2}(v)^5 f_b^1(v)+\frac{2}{5v^3}\int_0^v (v')^5 f_b^1(v')dv'\\ &- v(v)^2 f_b^1(v)+\int_v^\infty (v')^2 f_b^1(v')dv'\\ &+ \frac{v^3}{5}f_b^1(v)-\frac{3v^2}{5}\int_v^\infty f_b^1(v')dv'\\ &= \frac{2}{5v^3}\int_0^v (v')^5 f_b^1(v')dv'+\int_v^\infty (v')^2 f_b^1(v')dv'-\frac{3v^2}{5}\int_v^\infty f_b^1(v')dv'\end{aligned}$$

$$\begin{aligned}6\frac{\partial^2 h_b^1(v)}{\partial v^2} &= \frac{2}{5v^3}(v)^5 f_b^1(v)-\frac{6}{5v^4}\int_0^v (v')^5 f_b^1(v')dv'\\ &- (v)^2 f_b^1(v)\\ &+ \frac{3v^2}{5}f_b^1(v)-\frac{6v}{5}\int_v^\infty f_b^1(v')dv'\\ &= -\frac{6}{5v^4}\int_0^v (v')^5 f_b^1(v')dv'-\frac{6v}{5}\int_v^\infty f_b^1(v')dv'\end{aligned}$$

$$\frac{\partial^2 h_b^1(v)}{\partial v^2}=-\frac{1}{5v^4}\int_0^v (v')^5 f_b^1(v')dv'-\frac{v}{5}\int_v^\infty f_b^1(v')dv' \qquad (104)$$

$$\begin{aligned}\frac{g_b(v,\theta)}{\cos\theta} &= g_b^1(v)\\ &= -\frac{1}{3}\left[\int_0^v \frac{(v')^3}{v^2}f_b^1(v')dv'+\int_v^\infty v f_b^1(v')dv'\right] \qquad (105)\end{aligned}$$

$$\frac{\partial g_b^1}{\partial v} = \frac{2}{3v^3}\int_0^v (v')^3 f_b^1(v')dv' - \frac{1}{3}\int_v^\infty f_b^1(v')dv' \tag{106}$$

$$D_{cvv}^{a/b} = A\frac{\partial^2 h_b^1}{\partial v^2}\cos\theta$$

$$F_{cv}^{a/b} = A\frac{\partial g_b^1}{\partial v}\cos\theta$$

where $A = -4\pi\Gamma^{a/b}/n_b$.

$$S_{cv}^{a/b} = -D_{cvv}^{a/b}\frac{\partial f_a}{\partial v} + F_{cv}^{a/b} f_a = -A\frac{\partial^2 h_b^1}{\partial v^2}\cos\theta\frac{\partial f_a}{\partial v} + A\frac{\partial g_b^1}{\partial v}\cos\theta f_a$$

$$\begin{aligned}
\frac{S_{cv}^{a/b}}{f_{am}\cos\theta} &= A\frac{\partial^2 h_b^1}{\partial v^2}\left(\frac{v}{v_{ta}^2}\right) + A\frac{\partial g_b^1}{\partial v} \\
&= -A\frac{1}{5v^4}\left(\frac{v}{v_{ta}^2}\right)\int_0^v (v')^5 f_b^1(v')dv' - A\frac{v}{5}\left(\frac{v}{v_{ta}^2}\right)\int_v^\infty f_b^1(v')dv' \\
&+ A\frac{2}{3v^3}\int_0^v (v')^3 f_b^1(v')dv' - \frac{1}{3}A\int_v^\infty f_b^1(v')dv' \\
&= -A\frac{1}{5v^3 v_{ta}^2}\int_0^v (v')^5 f_b^1(v')dv' - A\frac{v^2}{5v_{ta}^2}\int_v^\infty f_b^1(v')dv' \\
&+ A\frac{2}{3v^3}\int_0^v (v')^3 f_b^1(v')dv' - \frac{1}{3}A\int_v^\infty f_b^1(v')dv' \\
&= -A\int_0^v \left[\frac{(v')^5}{5v_{ta}^2 v^3} - \frac{2(v')^3}{3v^3}\right] f_b^1(v')dv' - A\int_v^\infty \left(\frac{v^2}{5v_{ta}^2} + \frac{1}{3}\right) f_b^1(v')dv'
\end{aligned} \tag{107}$$

This is the Eq.(35a) in Karney1986 paper.

Next calculate $S_{c\theta}^{a/b}$

$$S_{c\theta}^{a/b} = -D_{c\theta v}^{a/b}\frac{\partial f_a}{\partial v} + F_{c\theta}^{a/b} f_a \tag{108}$$

$$D_{c\theta v}^{a/b} = A\left(\frac{1}{v}\frac{\partial^2 h_b}{\partial\theta\partial v} - \frac{1}{v^2}\frac{\partial h_b}{\partial\theta}\right) \tag{109}$$

$$F_{c\theta}^{a/b} = A\frac{1}{v}\frac{\partial g_b}{\partial\theta} \tag{110}$$

$$\begin{aligned}
\frac{\partial h_b}{\partial\theta} &= \frac{1}{6}\left[\int_0^v (v')^3 f_b^1(v')dv' - \frac{1}{5v^2}\int_0^v (v')^5 f_b^1(v')dv' \right.\\
&+ \left. v\int_v^\infty (v')^2 f_b^1(v')dv' - \frac{v^3}{5}\int_v^\infty f_b^1(v')dv'\right](-\sin\theta)
\end{aligned} \tag{111}$$

$$\frac{\partial g_b}{\partial\theta} = -\frac{1}{3}\left[\int_0^v \frac{(v')^3}{v^2} f_b^1(v')dv' + \int_v^\infty v f_b^1(v')dv'\right](-\sin\theta) \tag{112}$$

$$\frac{\partial^2 h_b}{\partial v\partial\theta} = \left[\frac{1}{15v^3}\int_0^v (v')^5 f_b^1(v')dv' + \frac{1}{6}\int_v^\infty (v')^2 f_b^1(v')dv' - \frac{v^2}{10}\int_v^\infty f_b^1(v')dv'\right](-\sin\theta) \tag{113}$$

$$\frac{D_{c\theta v}^{a/b}}{A(-\sin\theta)} = \frac{1}{10v^4}\int_0^v (v')^5 f_b^1(v')dv' - \frac{1}{6v^2}\int_0^v (v')^3 f_b^1(v')dv' - \frac{v}{15}\int_v^\infty f_b^1(v')dv' \tag{114}$$

$$\begin{aligned}
\frac{S_{c\theta}^{a/b}}{A f_{am}(-\sin\theta)} &= \frac{1}{10v^4}\frac{v}{v_{ta}^2}\int_0^v (v')^5 f_b^1(v')dv' - \frac{v}{v_{ta}^2}\frac{1}{6v^2}\int_0^v (v')^3 f_b^1(v')dv' \\
&- \frac{v}{v_{ta}^2}\frac{v}{15}\int_v^\infty f_b^1(v')dv' - \int_0^v \frac{1}{3v}\frac{(v')^3}{v^2} f_b^1(v')dv' - \frac{1}{3v}\int_v^\infty v f_b^1(v')dv' \\
&= \int_0^v \left(\frac{(v')^5}{10v_{ta}^2 v^3} - \frac{(v')^3}{6v_{ta}^2 v} - \frac{(v')^3}{3v^3}\right) f_b^1(v')dv' - \left(\frac{v^2}{15v_{ta}^2} + \frac{1}{3}\right)\int_v^\infty f_b^1(v')dv'
\end{aligned} \tag{115}$$

This is the Eq.(35b) in Karney1986 paper.

Next calculate $C(f_{am}, f^1(v)\cos\theta)$

$$\frac{C(f_{am}, f^1(v)\cos\theta)}{f_{am}\cos\theta} = \frac{-\nabla\cdot\boldsymbol{S}_c^{a/b}}{f_{am}\cos\theta} \tag{116}$$

$$\nabla\cdot\boldsymbol{S}_c^{a/b} = \frac{1}{v^2}\frac{\partial}{\partial v}v^2 S_{cv}^{a/b} + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_{c\theta}^{a/b} \tag{117}$$

$$\begin{aligned}
&\frac{1}{-Af_{am}\cos\theta}\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_{c\theta}^{a/b} \qquad (118)\\
&=\frac{1}{\cos\theta}\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\left[\sin^2\theta\frac{S_{c\theta}^{a/b}}{-Af_{am}\sin\theta}\right]\\
&=\frac{S_{c\theta}^{a/b}}{-Af_{am}\sin\theta}\frac{1}{\cos\theta}\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}[\sin^2\theta]\\
&=\frac{S_{c\theta}^{a/b}}{-Af_{am}\sin\theta}\frac{1}{\cos\theta}\frac{2\cos\theta}{v}\\
&=\frac{S_{c\theta}^{a/b}}{-Af_{am}\sin\theta}\frac{2}{v}\\
&=\int_0^v\left(\frac{(v')^5}{5v_{ta}^2v^4}-\frac{(v')^3}{3v_{ta}^2v^2}-\frac{2(v')^3}{3v^4}\right)f_b^1(v')dv'-\left(\frac{2v}{15v_{ta}^2}+\frac{2}{3v}\right)\int_v^\infty f_b^1(v')dv' \qquad (119)
\end{aligned}$$

$$\frac{S_{cv}^{a/b}}{-Af_{am}\cos\theta}=\int_0^v\left[\frac{(v')^5}{5v_{ta}^2v^3}-\frac{2(v')^3}{3v^3}\right]f_b^1(v')dv'+\int_v^\infty\left(\frac{v^2}{5v_{ta}^2}+\frac{1}{3}\right)f_b^1(v')dv' \qquad (120)$$

$$\frac{S_{c\theta}^{a/b}}{-Af_{am}\sin\theta}=\int_0^v\left(\frac{(v')^5}{10v_{ta}^2v^3}-\frac{(v')^3}{6v_{ta}^2v}-\frac{(v')^3}{3v^3}\right)f_b^1(v')dv'-\left(\frac{v^2}{15v_{ta}^2}+\frac{1}{3}\right)\int_v^\infty f_b^1(v')dv' \qquad (121)$$

$$\begin{aligned}
&\frac{1}{f_{am}(-A)\cos\theta}\frac{1}{v^2}\frac{\partial}{\partial v}v^2S_{cv}^{a/b}\\
&=\frac{1}{f_{am}}\frac{1}{v^2}\frac{\partial}{\partial v}\frac{v^2S_{cv}^{a/b}}{(-A)\cos\theta}\\
&=\frac{1}{v^2}\frac{1}{f_{am}}\frac{\partial}{\partial v}\left[\frac{v^2S_{cv}^{a/b}}{(-A)\cos\theta f_{am}}f_{am}\right]\\
&=\frac{1}{v^2}E \qquad (122)
\end{aligned}$$

$$E=\frac{1}{f_{am}}\frac{\partial}{\partial v}\left[\frac{v^2S_{cv}^{a/b}}{(-A)\cos\theta f_{am}}f_{am}\right]=\left(-\frac{v}{v_{ta}^2}\right)\frac{v^2S_{cv}^{a/b}}{-A\cos\theta f_{am}}+\frac{\partial}{\partial v}\left(\frac{v^2S_{cv}^{a/b}}{-Af_{am}\cos\theta}\right) \qquad (123)$$

First term of Eq.(123)

$$\begin{aligned}
\frac{v^2S_{cv}^{a/b}}{-Af_{am}\cos\theta} &= \int_0^v\left[\frac{(v')^5}{5v_{ta}^2v}-\frac{2(v')^3}{3v}\right]f_b^1(v')dv'+\int_v^\infty\left(\frac{v^4}{5v_{ta}^2}+\frac{v^2}{3}\right)f_b^1(v')dv'\\
&= \frac{1}{5v_{ta}^2v}\int_0^v(v')^5f_b^1(v')dv'-\frac{2}{3v}\int_0^v(v')^3f_b^1(v')dv'+\left(\frac{v^4}{5v_{ta}^2}+\frac{v^2}{3}\right)\int_v^\infty f_b^1(v')dv'
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial}{\partial v}\left(\frac{v^2S_{cv}^{a/b}}{-Af_{am}\cos\theta}\right) \qquad (124)\\
&=\frac{1}{5v_{ta}^2v}(v)^5f_b^1(v)-\frac{1}{5v_{ta}^2v^2}\int_0^v(v')^5f_b^1(v')dv'\\
&-\frac{2}{3v}(v)^3f_b^1(v)+\frac{2}{3v^2}\int_0^v(v')^3f_b^1(v')dv'\\
&-\left(\frac{v^4}{5v_{ta}^2}+\frac{v^2}{3}\right)f_b^1(v)+\left(\frac{4v^3}{5v_{ta}^2}+\frac{2v}{3}\right)\int_v^\infty f_b^1(v')dv'\\
&=-f_b^1(v)-\frac{1}{5v_{ta}^2v^2}\int_0^v(v')^5f_b^1(v')dv'+\frac{2}{3v^2}\int_0^v(v')^3f_b^1(v')dv'\\
&+\left(\frac{4v^3}{5v_{ta}^2}+\frac{2v}{3}\right)\int_v^\infty f_b^1(v')dv'
\end{aligned}$$

First term of Eq.(123)

$$\begin{aligned}
&\frac{1}{v^2}\left(-\frac{v}{v_{ta}^2}\right)\frac{v^2S_{cv}^{a/b}}{-Af_{am}\cos\theta} \qquad (125)\\
&=\int_0^v\left[-\frac{(v')^5}{5v_{ta}^4v^2}+\frac{2(v')^3}{3v_{ta}^2v^2}\right]f_b^1(v')dv'+\int_v^\infty\left(-\frac{v^3}{5v_{ta}^4}-\frac{v}{3v_{ta}^2}\right)f_b^1(v')dv
\end{aligned}$$

Second term of Eq.(123)

$$\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2 S_{cv}^{a/b}}{-Af_{am}\cos\theta}\right) \tag{126}$$
$$= -f_b^1(v) - \frac{1}{5v_{ta}^2 v^4}\int_0^v (v')^5 f_b^1(v')dv' + \frac{2}{3v^4}\int_0^v (v')^3 f_b^1(v')dv'$$
$$+\left(\frac{4v}{5v_{ta}^2}+\frac{2}{3v}\right)\int_v^\infty f_b^1(v')dv'$$

The sum of Eqs.(119)(125)(126)

$$= -f_b^1(v) - \frac{1}{5v_{ta}^2 v^4}\int_0^v (v')^5 f_b^1(v')dv' + \frac{2}{3v^4}\int_0^v (v')^3 f_b^1(v')dv'$$
$$+\left(\frac{4v}{5v_{ta}^2}+\frac{2}{3v}\right)\int_v^\infty f_b^1(v')dv'$$
$$+\int_0^v\left[-\frac{(v')^5}{5v_{ta}^4 v^2}+\frac{2(v')^3}{3v_{ta}^2 v^2}\right]f_b^1(v')dv' + \int_v^\infty\left(-\frac{v^3}{5v_{ta}^4}-\frac{v}{3v_{ta}^2}\right)f_b^1(v')dv'$$
$$+\int_0^v\left(\frac{(v')^5}{5v_{ta}^2 v^4}-\frac{(v')^3}{3v_{ta}^2 v^2}-\frac{2(v')^3}{3v^4}\right)f_b^1(v')dv' - \left(\frac{2v}{15v_{ta}^2}+\frac{2}{3v}\right)\int_v^\infty f_b^1(v')dv'$$

Coefficient of $\int_0^v f_b^1(v')dv'$

$$-\frac{1}{5v_{ta}^2 v^4}(v')^5 + \frac{2}{3v^4}(v')^3 + \left[-\frac{(v')^5}{5v_{ta}^4 v^2}+\frac{2(v')^3}{3v_{ta}^2 v^2}\right] + \frac{(v')^5}{5v_{ta}^2 v^4} - \frac{(v')^3}{3v_{ta}^2 v^2} - \frac{2(v')^3}{3v^4}$$
$$= -\frac{(v')^5}{5v_{ta}^4 v^2} + \frac{(v')^3}{3v_{ta}^2 v^2}$$
$$= \frac{v'^2}{v_{ta}^2}\left(-\frac{v'^3}{5v_{ta}^2 v^2}+\frac{v'}{3v^2}\right)$$

Coefficient of $\int_v^\infty f_b^1(v')dv'$

$$\frac{4v}{5v_{ta}^2}+\frac{2}{3v}-\frac{v^3}{5v_{ta}^4}-\frac{v}{3v_{ta}^2}-\left(\frac{2v}{15v_{ta}^2}+\frac{2}{3v}\right)$$
$$= \frac{v}{3v_{ta}^2} - \frac{v^3}{5v_{ta}^4} = \frac{v'^2}{v_{ta}^2}\left(\frac{v}{3v'^2}-\frac{v^3}{5v_{ta}^2 v'^2}\right)$$

9 Another presentation of collision term

This presention of collision term was used in Rosenbluth's original paper[5], and is also used in serveral numerical codes, thus, is needed to be recognized. In this presentation, the collision term is given by

$$\left(\frac{\partial f_a}{\partial t}\right)_c = \frac{\Gamma^{a/b}}{n_b}\left[-\frac{\partial}{\partial v_i}\left(f_a\frac{\partial h}{\partial v_i}\right)+\frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(f_a\frac{\partial^2 g}{\partial v_i \partial v_j}\right)\right] \tag{127}$$

where the usual convention of summing over repeated indices i and j is used, and

$$h(\boldsymbol{v}) = \frac{m_a+m_b}{m_b}\int d^3\boldsymbol{v}'\frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}, \tag{128}$$

$$g(\boldsymbol{v}) = \int d^3\boldsymbol{v}' f_b(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|, \tag{129}$$

It is obvious that

$$\frac{\partial}{\partial v_i}\left(f_a\frac{\partial h}{\partial v_i}\right) = \nabla\cdot(f_a\nabla h) \tag{130}$$

and it can be proved that (to be proved by me)

$$\frac{\partial^2}{\partial v_i \partial v_j}\left(f_a\frac{\partial^2 g}{\partial v_i \partial v_j}\right) = \nabla\cdot[\nabla\cdot(f_a\nabla\nabla g)] \tag{131}$$

Using these we can recover the corresponding collision flux in Eq. (127),

$$\left(\frac{\partial f}{\partial t}\right)_c = -\nabla\cdot\boldsymbol{S}$$

with

$$
\begin{aligned}
\frac{n_b}{\Gamma^{a/b}}\boldsymbol{S} &= (\nabla h) f_a - \frac{1}{2}\nabla\cdot(f_a\nabla\nabla g) \\
&= (\nabla h) f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a - \frac{1}{2} f_a\nabla\cdot(\nabla\nabla g) \\
&= (\nabla h) f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a - \frac{1}{2} f_a\nabla(\nabla^2 g) \\
&= (\nabla h) f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a - \frac{m_b}{m_a+m_b} f_a\nabla h \\
&= \frac{m_a}{m_a+m_b}(\nabla h) f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a,
\end{aligned} \tag{132}
$$

where use has been made of

$$
\nabla^2 g(\boldsymbol{v}) = 2\frac{m_b}{m_a+m_b} h(\boldsymbol{v}) \tag{133}
$$

In Karney's notation

$$
\boldsymbol{S}_c^{a/b}(\boldsymbol{v}) = \boldsymbol{F}_c^{a/b} f_a(\boldsymbol{v}) - \boldsymbol{D}_c^{a/b}\cdot\nabla f_a(\boldsymbol{v}) \tag{134}
$$

where

$$
\begin{aligned}
\boldsymbol{F}_c^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\nabla\phi_b(\boldsymbol{v}), \\
\boldsymbol{D}_c^{a/b} &= -\frac{4\pi\Gamma^{a/b}}{n_b}\nabla\nabla\psi_b(\boldsymbol{v}),
\end{aligned}
$$

$$
\phi_b(\boldsymbol{v}) = -\frac{1}{4\pi}\int d\boldsymbol{v}'\frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}
$$

$$
\psi_b(\boldsymbol{v}) = -\frac{1}{8\pi}\int d\boldsymbol{v}' f_b(\boldsymbol{v}')|\boldsymbol{v}-\boldsymbol{v}'|
$$

$$
\nabla^2\phi_b(\boldsymbol{v}) = f_b(\boldsymbol{v})
$$

$$
\nabla^2\psi_b(\boldsymbol{v}) = \phi_b(\boldsymbol{v})
$$

then one knows that

$$
\phi_b(\boldsymbol{v}) = -\frac{1}{4\pi}\frac{m_b}{m_a+m_b} h(\boldsymbol{v})
$$

$$
\psi_b(\boldsymbol{v}) = -\frac{1}{8\pi} g(\boldsymbol{v})
$$

Using this, one can prove that Karney's presentation agrees with Rosenbluth's original results,

$$
\begin{aligned}
\boldsymbol{S}_c^{a/b}(\boldsymbol{v}) &= \boldsymbol{F}_c^{a/b} f_a(\boldsymbol{v}) - \boldsymbol{D}_c^{a/b}\cdot\nabla f_a(\boldsymbol{v}) \\
&= -\frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\nabla\phi_b(\boldsymbol{v}) f_a(\boldsymbol{v}) + \frac{4\pi\Gamma^{a/b}}{n_b}\nabla\nabla\psi_b(\boldsymbol{v})\cdot\nabla f_a(\boldsymbol{v}) \\
&= \frac{4\pi\Gamma^{a/b}}{n_b}\frac{m_a}{m_b}\frac{1}{4\pi}\frac{m_b}{m_a+m_b}(\nabla h) f_a(\boldsymbol{v}) - \frac{4\pi\Gamma^{a/b}}{n_b}\frac{1}{8\pi}\nabla\nabla g(\boldsymbol{v})\cdot\nabla f_a(\boldsymbol{v}) \\
&= \frac{\Gamma^{a/b}}{n_b}\frac{m_a}{m_a+m_b}(\nabla h) f_a(\boldsymbol{v}) - \frac{\Gamma^{a/b}}{n_b}\frac{1}{2}\nabla\nabla g(\boldsymbol{v})\cdot\nabla f_a(\boldsymbol{v})
\end{aligned} \tag{135}
$$

This agrees with Eq. (132).

10 Calculation of various coefficients

$$
\begin{aligned}
\frac{n_b}{\Gamma^{a/b}}\left(\frac{\partial f}{\partial t}\right)_c &= -\nabla\cdot\left[\frac{n_b}{\Gamma^{a/b}}\boldsymbol{S}\right] \\
&= \frac{1}{v^2}\frac{\partial G_a}{\partial v} + \frac{1}{v^2\sin\theta}\frac{\partial H_a}{\partial\theta}
\end{aligned}
$$

with

$$
G_a = -v^2\frac{n_b}{\Gamma^{a/b}} S_v \tag{136}
$$

$$
H_a = -v\sin\theta\frac{n_b}{\Gamma^{a/b}} S_\theta \tag{137}
$$

$$
\begin{aligned}
\frac{n_b}{\Gamma^{a/b}}\boldsymbol{S} &= (\nabla h) f_a - \frac{1}{2}\nabla\cdot(f_a\nabla\nabla g) \\
&= (\nabla h) f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a - \frac{1}{2} f_a\nabla(\nabla^2 g)
\end{aligned} \tag{138}
$$

From Eq. (138), one gets

$$\begin{aligned}\frac{n_b}{\Gamma^{a/b}}S_v &= \frac{\partial h}{\partial v}f_a - \frac{1}{2}[(\nabla\nabla g)\cdot\nabla f_a]_v - \frac{1}{2}f_a\frac{\partial}{\partial v}[\nabla^2 g] \\ &= \frac{\partial h}{\partial v}f_a - \frac{1}{2}\frac{\partial^2 g}{\partial v^2}\frac{\partial f_a}{\partial v} - \frac{1}{2}\left(\frac{1}{v}\frac{\partial^2 g}{\partial\theta\partial v} - \frac{1}{v^2}\frac{\partial g}{\partial\theta}\right)\frac{1}{v}\frac{\partial f_a}{\partial\theta} \\ &- \frac{1}{2}f_a\frac{\partial}{\partial v}\left[\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial g}{\partial v}\right) + \frac{1}{v^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right)\right]\end{aligned}$$

Using this, G_a can be written in the form

$$G_a = A_a f_a + B_a\frac{\partial f_a}{\partial v} + C_a\frac{\partial f_a}{\partial\theta}, \tag{139}$$

with

$$\begin{aligned}A_a &= -v^2\frac{\partial h}{\partial v} + v^2\frac{1}{2}\frac{\partial}{\partial v}[\nabla^2 g] \\ &= \frac{v^2}{2}\frac{\partial^3 g}{\partial v^3} + v\frac{\partial^2 g}{\partial v^2} - \frac{\partial g}{\partial v} - v^2\frac{\partial h}{\partial v} - \frac{1}{v}\frac{\partial^2 g}{\partial\theta^2} + \frac{1}{2}\frac{\partial^3 g}{\partial\theta^2\partial v} - \frac{\cos\theta}{\sin\theta}\frac{1}{v}\frac{\partial g}{\partial\theta} + \frac{1}{2}\frac{\cos\theta}{\sin\theta}\frac{\partial^2 g}{\partial\theta\partial v},\end{aligned} \tag{140}$$

$$B_a = \frac{v^2}{2}\frac{\partial^2 g}{\partial v^2}, \tag{141}$$

$$C_a = -\frac{1}{2v}\frac{\partial g}{\partial\theta} + \frac{1}{2}\frac{\partial^2 g}{\partial\theta\partial v}. \tag{142}$$

Eqs. (140), (141), and (142) agree respectively with Eqs. (8), (9), and (10) in Ref.[6].

$$H_a = -v\sin\theta\frac{n_b}{\Gamma^{a/b}}S_\theta$$

$$H_a = D_a f_a + E_a\frac{\partial f_a}{\partial v} + F_a\frac{\partial f_a}{\partial\theta} \tag{143}$$

$$\begin{aligned}v\sin\theta\frac{n_b}{\Gamma^{a/b}}S_\theta &= v\sin\theta\frac{1}{v}\frac{\partial h}{\partial\theta}f_a - v\sin\theta\frac{1}{2}[(\nabla\nabla g)\cdot\nabla f_a]_\theta - v\sin\theta\frac{1}{2}f_a\frac{1}{v}\frac{\partial}{\partial\theta}[\nabla^2 g] \\ &= v\sin\theta\frac{1}{v}\frac{\partial h}{\partial\theta}f_a - v\sin\theta\frac{1}{2}\left\{\left(\frac{1}{v}\frac{\partial^2 g}{\partial\theta\partial v} - \frac{1}{v^2}\frac{\partial g}{\partial\theta}\right)\frac{\partial f_a}{\partial v} + \left(\frac{1}{v}\frac{\partial g}{\partial v} + \frac{1}{v^2}\frac{\partial^2 g}{\partial\theta^2}\right)\frac{1}{v}\frac{\partial f_a}{\partial\theta}\right\} \\ &- v\sin\theta\frac{1}{2}f_a\frac{1}{v}\frac{\partial}{\partial\theta}[\nabla^2 g]\end{aligned}$$

$$D_a = -\sin\theta\frac{\partial h}{\partial\theta} + \sin\theta\frac{1}{2}\frac{\partial}{\partial\theta}[\nabla^2 g] \tag{144}$$

$$E_a = \sin\theta\frac{1}{2}\left(\frac{\partial^2 g}{\partial\theta\partial v} - \frac{1}{v}\frac{\partial g}{\partial\theta}\right) \tag{145}$$

$$F_a = \sin\theta\frac{1}{2}\left(\frac{1}{v}\frac{\partial g}{\partial v} + \frac{1}{v^2}\frac{\partial^2 g}{\partial\theta^2}\right) \tag{146}$$

Eqs. (144), (145), and (146) agree respectively with Eqs. (11), (12), and (13) in Ref.[6].

11 A third presentation

Using the relation,

$$\nabla^2 g(\boldsymbol{v}) = 2\frac{m_b}{m_a + m_b}h(\boldsymbol{v}) \tag{147}$$

to eliminate h in Eq. (132), we obtain

$$\frac{n_b}{\Gamma^{a/b}}\boldsymbol{S} = \frac{1}{2}\frac{m_a}{m_b}\nabla\big(\nabla^2 g(\boldsymbol{v})\big)f_a - \frac{1}{2}(\nabla\nabla g)\cdot\nabla f_a \tag{148}$$

This form [Eq.(148)] involves only one potential g, and is convenient in some cases. CQL3D use this presentation (in the relativistic form) to express relativistic collision term.

Bibliography

[1] Nathaniel J. Fisch. Theory of current drive in plasmas. *Rev. Mod. Phys.*, 59(1):175–234, Jan 1987.

[2] N. J. Fisch. Current generation in a relativistic plasma. *Phys. Rev. A*, 24(6):3245–3248, Dec 1981.

[3] Y. R. Lin-Liu, V. S. Chan, and R. Prater. Electron cyclotron current drive efficiency in general tokamak geometry. *Phys. Plasmas*, 10(10):4064–4071, 2003.

[4] Charles F. F. Karney. Fokker-planck and quasilinear codes. *Comp. Phys. Rep.*, 4:183–244, 1986.

[5] Marshall N. Rosenbluth, William M. MacDonald, and David L. Judd. Fokker-planck equation for an inverse-square force. *Phys. Rev.*, 107(1):1–6, Jul 1957.

[6] M. G. McCoy, A. A. Mirin, and J. Killeen. Fppac: A two-dimensional multispecies nonlinear fokker-planck package. *Computer Physics Communications*, 24(1):37 – 61, 1981.

12 Manuscript

$$\nabla\cdot\boldsymbol{S}^{a/b}=\frac{1}{v^2}\frac{\partial}{\partial v}v^2S_v+\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_\theta$$

$$S_v=-\Gamma^{a/b}\frac{v_{tb}^2}{v^3}\frac{\partial f_a}{\partial v}-\Gamma^{a/b}\frac{m_a}{m_b}\frac{1}{v^2}f_a$$

$$S_\theta=-\Gamma^{a/b}\frac{1}{2v}\left(1-\frac{v_{tb}^2}{v^2}\right)\frac{1}{v}\frac{\partial f_a}{\partial\theta}$$

$$C_{ab}[f_a,f_b]=-\Gamma_aZ_b^2\frac{\partial}{\partial\boldsymbol{v}}\cdot\left[\frac{m_a}{m_b}\frac{\partial h_b}{\partial\boldsymbol{v}}f_a-\frac{1}{2}\frac{\partial^2g_b}{\partial\boldsymbol{v}\partial\boldsymbol{v}}\cdot\frac{\partial f_a}{\partial\boldsymbol{v}}\right]$$

$$\Gamma_a=\frac{4\pi z_a^2e^4\ln\Lambda}{m_a^2}$$

$$c_{ab}=2\pi e_a^2e_b^2\ln\Lambda=\frac{\Gamma_az_b^2m_a^2}{2}$$

$$\frac{c_{ab}}{m_a^2}(\nabla\nabla g_b(\boldsymbol{v}))\cdot\nabla f_a(\boldsymbol{v})-\frac{2c_{ab}}{m_am_b}\nabla h_b(\boldsymbol{v})f_a(\boldsymbol{v})=$$

$$-\Gamma_az_b^2\left[-\frac{1}{2}(\nabla\nabla g_b(\boldsymbol{v}))\cdot\nabla f_a(\boldsymbol{v})\right]-\Gamma_az_b^2\frac{m_a}{m_b}\nabla h_b(\boldsymbol{v})f_a(\boldsymbol{v})$$

Proof of Eq.(9)

$$\begin{aligned}\nabla^2h_b(\boldsymbol{v}) &= \nabla^2\int d\boldsymbol{v'}\frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|}\\ &= -\frac{1}{4\pi}\int d\boldsymbol{v'}f_b(\boldsymbol{v}')\nabla^2\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|}\\ &= -\frac{1}{4\pi}\int d\boldsymbol{v'}f_b(\boldsymbol{v}')4\pi\delta(\boldsymbol{v}-\boldsymbol{v}')\\ &= f_b(\boldsymbol{v})\end{aligned}$$

13 Proof

$$\nabla u=\frac{\boldsymbol{u}}{u}\tag{149}$$

$$\nabla\boldsymbol{u}=\boldsymbol{I}\tag{150}$$

$$\nabla\left(\frac{1}{u}\right)=-\frac{1}{u^2}\nabla u=-\frac{\boldsymbol{u}}{u^3}\tag{151}$$

$$\nabla'\left(\frac{1}{u}\right)=-\frac{1}{u^2}\nabla'u=\frac{\boldsymbol{u}}{u^3}$$

$$\nabla^2\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|}=-4\pi\delta(\boldsymbol{v}-\boldsymbol{v}')$$

$$\nabla\nabla u=\nabla\left(\frac{\boldsymbol{u}}{u}\right)=\frac{1}{u}\nabla\boldsymbol{u}+\nabla\left(\frac{1}{u}\right)\boldsymbol{u}=\frac{\boldsymbol{I}}{u}-\frac{\boldsymbol{u}\boldsymbol{u}}{u^3}$$

$$\nabla'\nabla u=\nabla'\left(\frac{\boldsymbol{u}}{u}\right)=\frac{1}{u}\nabla'\boldsymbol{u}+\nabla'\left(\frac{1}{u}\right)\boldsymbol{u}=\frac{-\boldsymbol{I}}{u}+\frac{\boldsymbol{u}\boldsymbol{u}}{u^3}$$

Proof of Eq.(149):

$$\nabla u=\nabla\sqrt{(\boldsymbol{v}-\boldsymbol{v}')\cdot(\boldsymbol{v}-\boldsymbol{v}')}=\frac{\nabla[(\boldsymbol{v}-\boldsymbol{v}')\cdot(\boldsymbol{v}-\boldsymbol{v}')]}{2\sqrt{(\boldsymbol{v}-\boldsymbol{v}')\cdot(\boldsymbol{v}-\boldsymbol{v}')}}=\frac{2(\boldsymbol{v}-\boldsymbol{v}')\cdot\nabla(\boldsymbol{v}-\boldsymbol{v}')}{2u}=\frac{\boldsymbol{u}\cdot\nabla\boldsymbol{u}}{u}=\frac{\boldsymbol{u}}{u}$$

Proof of Eq.(150)

$$\nabla \boldsymbol{u} = \nabla u_x \boldsymbol{e}_x + \nabla u_y \boldsymbol{e}_y + \nabla u_z \boldsymbol{e}_z = \boldsymbol{e}_x \boldsymbol{e}_x + \boldsymbol{e}_y \boldsymbol{e}_y + \boldsymbol{e}_z \boldsymbol{e}_z = \boldsymbol{I}$$

$$\boldsymbol{r} \cdot \nabla \boldsymbol{r} = x\frac{\partial}{\partial x}\boldsymbol{r} + y\frac{\partial}{\partial y}\boldsymbol{r} + z\frac{\partial}{\partial z}\boldsymbol{r} = \boldsymbol{r}$$

$$\nabla v = \frac{\boldsymbol{v}}{v}$$

$$\frac{d|\boldsymbol{v}-\boldsymbol{v}'|}{d\boldsymbol{v}'} = \frac{d\sqrt{(v_x - v_x')^2 + (v_y - v_y')^2 + (v_z - v_z')^2}}{d\boldsymbol{v}'} = x\frac{v_x' - v_x}{u} + y\frac{v_y' - v_y}{u} + z\frac{v_z' - v_z}{u}$$

$$= \frac{\boldsymbol{v}' - \boldsymbol{v}}{u}$$

$$\frac{d\sqrt{(\boldsymbol{v}-\boldsymbol{v}')\cdot(\boldsymbol{v}-\boldsymbol{v}')}}{d\boldsymbol{v}'} = \frac{-2(\boldsymbol{v}-\boldsymbol{v}')}{2\sqrt{(\boldsymbol{v}-\boldsymbol{v}')\cdot(\boldsymbol{v}-\boldsymbol{v}')}} = \frac{\boldsymbol{v}'-\boldsymbol{v}}{u}$$

Use the following relation

$$\nabla\nabla u = \frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{u}\boldsymbol{u}}{u^3} \tag{152}$$

or it can be written as

$$\frac{\partial^2 u}{\partial \boldsymbol{v} \partial \boldsymbol{v}} = \frac{\boldsymbol{I}}{u} - \frac{\boldsymbol{u}\boldsymbol{u}}{u^3}$$

I prefer the former representation.

$$\frac{\partial^2}{\partial v_i \partial v_j}\left(f\frac{\partial^2 g}{\partial v_i \partial v_j}\right) = \nabla \cdot [\nabla \cdot (f \nabla\nabla g)]$$

Proof:

$$\begin{gathered}
\frac{\partial^2}{\partial v_x \partial v_x}\left(f\frac{\partial^2 g}{\partial v_x \partial v_x}\right) + \frac{\partial^2}{\partial v_x \partial v_y}\left(f\frac{\partial^2 g}{\partial v_x \partial v_y}\right) + \frac{\partial^2}{\partial v_x \partial v_z}\left(f\frac{\partial^2 g}{\partial v_x \partial v_z}\right) \\
\frac{\partial^2}{\partial v_y \partial v_x}\left(f\frac{\partial^2 g}{\partial v_y \partial v_x}\right) + \frac{\partial^2}{\partial v_y \partial v_y}\left(f\frac{\partial^2 g}{\partial v_y \partial v_y}\right) + \frac{\partial^2}{\partial v_y \partial v_z}\left(f\frac{\partial^2 g}{\partial v_y \partial v_z}\right) \\
\frac{\partial^2}{\partial v_z \partial v_x}\left(f\frac{\partial^2 g}{\partial v_z \partial v_x}\right) + \frac{\partial^2}{\partial v_z \partial v_y}\left(f\frac{\partial^2 g}{\partial v_z \partial v_y}\right) + \frac{\partial^2}{\partial v_z \partial v_z}\left(f\frac{\partial^2 g}{\partial v_z \partial v_z}\right)
\end{gathered}$$

$$\begin{gathered}
\frac{\partial^2}{\partial v_x \partial v_x}\left(f\frac{\partial^2 g}{\partial v_x \partial v_x}\right) + 2\frac{\partial^2}{\partial v_x \partial v_y}\left(f\frac{\partial^2 g}{\partial v_x \partial v_y}\right) + 2\frac{\partial^2}{\partial v_x \partial v_z}\left(f\frac{\partial^2 g}{\partial v_x \partial v_z}\right) \\
+ \frac{\partial^2}{\partial v_y \partial v_y}\left(f\frac{\partial^2 g}{\partial v_y \partial v_y}\right) + 2\frac{\partial^2}{\partial v_y \partial v_z}\left(f\frac{\partial^2 g}{\partial v_y \partial v_z}\right) \\
+ \frac{\partial^2}{\partial v_z \partial v_z}\left(f\frac{\partial^2 g}{\partial v_z \partial v_z}\right)
\end{gathered}$$

to be continued.

14 Self-adjoint property of collision operator

We know that the linearized collision operator have the property

$$\int g C^l(h f_m) d^3\boldsymbol{u} = \int h C^l(g f_m) d^3\boldsymbol{u},$$

(I do not prove this.) In the following I want to investigate what this property means when $g = g(u)$ and $h = \chi(u)\cos\theta$.

$$\begin{aligned}
\int g C^l(h f_m) d^3\boldsymbol{u} &= 2\pi \int_0^\infty \int_0^{2\pi} g C^l(h f_m) u^2 \sin\theta d\theta du \\
&= 2\pi \int_0^\infty \int_0^{2\pi} g C^l(\chi(u)\cos\theta f_m) u^2 \sin\theta d\theta du \\
&= 2\pi \int_0^\infty g u^2 du \int_0^\pi \frac{C^l(\chi(u)\cos\theta f_m)}{\cos\theta} \cos\theta \sin\theta d\theta
\end{aligned} \tag{153}$$

Since

$$H(u) \equiv \frac{C^l(\chi(u)\cos\theta f_m)}{\cos\theta},$$

is independent of θ, thus we can take this term out of the θ integration in Eq. (153) giving

$$\begin{aligned}\int g C^l(h f_m) d^3\boldsymbol{u} &= 2\pi \int_0^\infty g u^2 H(u) du \int_0^\pi \cos\theta \sin\theta d\theta \\ &= 0\end{aligned}$$

stop* no need to proceed

15 To prove $C^{e/e}(\frac{v_\parallel}{v_{te}} f_{em}, f_{em}) + C^{e/e}(f_{em}, \frac{v_\parallel}{v_{te}} f_{em}) = 0$

We first consdier the $C^{e/e}(\frac{v_\parallel}{v_{te}} f_{em}, f_{em})$ term. For isotropic backgound distribution, the collision flux is written as

$$S_{cv}^{a/b} = -D_{cvv}^{a/b} \frac{\partial f_a}{\partial u} + F_{cv}^{a/b} f_a, \tag{154}$$

$$S_{c\theta}^{a/b} = -D_{c\theta\theta}^{a/b} \frac{1}{v} \frac{\partial f_a}{\partial \theta}, \tag{155}$$

The collision term is the divergence of the collision flux,

$$-C^{e/e}(\frac{v_\parallel}{v_{te}} f_{em}, f_{em}) = \nabla_v \cdot \boldsymbol{S} \tag{156}$$

$$\begin{aligned} &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2 S_{cv} + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta S_{c\theta}. \\ &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \left(-D_{cvv}^{a/b} \frac{\partial\left(\frac{v_\parallel}{v_{te}} f_{em}\right)}{\partial v} + F_{cv}^{a/b} \frac{v_\parallel}{v_{te}} f_{em}, \right) + \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta \left(-D_{c\theta\theta}^{a/b} \frac{1}{v} \frac{\partial \frac{v_\parallel}{v_{te}} f_{em}}{\partial\theta} \right) \\ &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \left(-D_{cvv}^{a/b} \frac{1}{v_{te}} \frac{\partial(v f_{em})}{\partial v} + F_{cv}^{a/b} \frac{v}{v_{te}} f_{em}, \right)\cos\theta + \frac{1}{v}(-D_{c\theta\theta}^{a/b} \frac{f_{em}}{v_{te}})(-2\cos\theta) \end{aligned} \tag{157}$$

Then

$$\begin{aligned}\frac{-C^{e/e}(\frac{v_\parallel}{v_{te}} f_{em}, f_{em})}{A f_{em}(v)\cos\theta} &= \frac{1}{v^2}\frac{\partial}{\partial v} v^2 \left(-\frac{D_{cvv}^{a/b}}{A}\left(\frac{1}{v_{te}} + \frac{v}{v_{te}}\left(-\frac{v}{v_{te}^2} \right) \right) + \frac{F_{cv}^{a/b}}{A}\frac{v}{v_{te}} \right) + \frac{2}{v}\frac{D_{c\theta\theta}^{a/b}}{A}\frac{1}{v_{te}} \\ &= \frac{1}{v^2}\frac{\partial}{\partial v}\left(-\frac{v^2}{v_{te}}\frac{D_{cvv}^{a/b}}{A} + \frac{v^4}{v_{te}^3}\frac{D_{cvv}^{a/b}}{A} + \frac{v^3}{v_{te}}\frac{F_{cv}^{a/b}}{A} \right) + \frac{2}{v v_{te}}\frac{D_{c\theta\theta}^{a/b}}{A},\end{aligned}$$

where $A = 4\pi\Gamma^{a/b}/n_b$. We have

$$\frac{D_{cvv}^{a/b}}{A} = \frac{1}{3}\left(\int_0^v \frac{(v')^4}{v^3} f_m(v') dv' + \int_v^\infty v' f_m(v') dv' \right) \tag{158}$$

$$\frac{F_{cv}^{a/b}}{A} = -\int_0^v \frac{(v')^2}{v^2} f_m(v') dv' \tag{159}$$

$$\frac{D_{c\theta\theta}^{a/b}}{A} = \frac{1}{3}\left[\int_0^v \frac{v'^2}{2v^3}(3v^2 - (v')^2) f_m(v') dv' + \int_v^\infty v' f_m(v') dv' \right] \tag{160}$$

Then

$$\begin{aligned}&-\frac{v^2}{v_{te}}\frac{D_{cvv}^{a/b}}{A} + \frac{v^4}{v_{te}^3}\frac{D_{cvv}^{a/b}}{A} + \frac{v^3}{v_{te}}\frac{F_{cv}^{a/b}}{A} \\ &= -\frac{v^2}{v_{te}}\frac{1}{3}\left(\int_0^v \frac{(v')^4}{v^3} f_m(v') dv' + \int_v^\infty v' f_m(v') dv' \right) + \frac{v^4}{v_{te}^3}\frac{1}{3}\left(\int_0^v \frac{(v')^4}{v^3} f_m(v') dv' + \int_v^\infty v' f_m(v') dv' \right) - \\ &\frac{v^3}{v_{te}}\int_0^v \frac{(v')^2}{v^2} f_m(v') dv' \\ &= -\frac{1}{3v_{te}}\frac{1}{v}\int_0^v (v')^4 f_m(v') dv' - \frac{1}{3v_{te}} v^2 \int_v^\infty v' f_m(v') dv' + \frac{1}{3v_{te}^3} v \int_0^v (v')^4 f_m(v') dv' + \frac{1}{3v_{te}^3} v^4 \int_v^\infty v' f_m(v') dv' - \\ &\frac{v}{v_{te}}\int_0^v (v')^2 f_m(v') dv'\end{aligned}$$

Then the partial derivativ of the above expression with respect to v is written as

$$\begin{aligned}&\frac{1}{3v_{te}}\frac{1}{v^2}\int_0^v (v')^4 f_m(v') dv' - \frac{1}{3v_{te}} v^3 f_m(v) - \frac{2}{3v_{te}} v \int_v^\infty v' f_m(v') dv' + \frac{1}{3v_{te}} v^3 f_m(v) \\ &+ \frac{1}{3v_{te}^3}\int_0^v (v')^4 f_m(v') dv' + \frac{v^5}{3v_{te}^3} f_m(v) + \frac{4v^3}{3v_{te}^3}\int_v^\infty v' f_m(v') dv' - \frac{v^5}{3v_{te}^3} f_m(v) - \frac{1}{v_{te}}\int_0^v (v')^2 f_m(v') dv' - \frac{v^3}{v_{te}} f_m(v)\end{aligned}$$

Multiplying the above expression by $1/v^2$ gives

$$\frac{1}{3v_{te}} \frac{1}{v^4} \int_0^v (v')^4 f_m(v') dv' - \frac{1}{3v_{te}} v f_m(v) - \frac{2}{3v_{te}} \frac{1}{v} \int_v^\infty v' f_m(v') dv' + \frac{1}{3v_{te}} v f_m(v)$$
$$+ \frac{1}{3v_{te}^3 v^2} \int_0^v (v')^4 f_m(v') dv' + \frac{v^3}{v_{te}^3} \frac{1}{3} f_m(v) + \frac{4v}{3v_{te}^3} \int_v^\infty v' f_m(v') dv' - \frac{v^3}{3v_{te}^3} f_m(v) - \frac{1}{v_{te} v^2} \int_0^v (v')^2 f_m(v') dv' - \frac{v}{v_{te}} f_m(v)$$

The terms that does not involve integration can be written as

$$-\frac{1}{v_{te}} \frac{1}{3} v f_m(v) + \frac{1}{v_{te}} \frac{1}{3} v f_m(v) + \frac{v^3}{v_{te}^3} \frac{1}{3} f_m(v) - \frac{v^3}{v_{te}^3} \frac{1}{3} f_m(v) - \frac{v}{v_{te}} f_m(v)$$
$$= -\frac{v}{v_{te}} f_m(v). \tag{161}$$

The terms that involves the integration $\int_0^v dv'$ are written as

$$\frac{1}{3v_{te}} \frac{1}{v^4} \int_0^v (v')^4 f_m(v') dv' + \frac{1}{3v_{te}^3 v^2} \int_0^v (v')^4 f_m(v') dv' - \frac{1}{v_{te} v^2} \int_0^v (v')^2 f_m(v') dv' + \frac{2}{v v_{te}} \frac{1}{3} \int_0^v \frac{v'^2}{2v^3} \big(3v^2 - (v')^2\big) f_m(v') dv'$$
$$= \int_0^v \left[\frac{1}{v_{te}} \frac{1}{3} \frac{1}{v^4} + \frac{1}{3v_{te}^3 v^2} - \frac{1}{v_{te} v^2 v'^2} + \frac{2}{v v_{te}} \frac{1}{3} \frac{1}{2v^3 v'^2} 3v^2 - \frac{2}{v v_{te}} \frac{1}{3} \frac{1}{2v^3} \right] (v')^4 f_m(v') dv'$$
$$= \int_0^v \left[\frac{1}{v_{te}} \frac{1}{3} \frac{1}{v^4} + \frac{1}{3v_{te}^3 v^2} - \frac{1}{v_{te} v^2 v'^2} + \frac{1}{v_{te} v^2 v'^2} - \frac{1}{v v_{te}} \frac{1}{3} \frac{1}{v^3} \right] (v')^4 f_m(v') dv'$$
$$= \int_0^v \left[\frac{1}{3v_{te}^3 v^2} \right] (v')^4 f_m(v') dv'$$

The terms that involves the integration $\int_v^\infty dv'$ are written as

$$-\frac{1}{v_{te}} \frac{2}{3v} \int_v^\infty v' f_m(v') dv' + \frac{4v}{3v_{te}^3} \int_v^\infty v' f_m(v') dv' + \frac{2}{v v_{te}} \frac{1}{3} \int_v^\infty v' f_m(v') dv'$$
$$= \int_v^\infty \left(\frac{4v}{3v_{te}^3} - \frac{1}{v_{te}} \frac{2}{3v} + \frac{2}{v v_{te}} \frac{1}{3} \right) v' f_m(v') dv'$$
$$= \int_v^\infty \left(\frac{4v}{3v_{te}^3} \right) v' f_m(v') dv'$$

Next consider the $C^{e/e}(f_{em}, \frac{v_\parallel}{v_{te}} f_{em})$ term. Using Eq. (34) in Karney's paper, we obtain

$$\frac{-C^{e/e}(f_{em}, \frac{v_\parallel}{v_{te}} f_{em})}{A f_{em}(v) \cos\theta} = \frac{-C^{e/e}(f_{em}, \frac{v}{v_{te}} f_{em} \cos\theta)}{A f_{em}(v) \cos\theta}$$
$$= f_m \frac{v}{v_t} + \int_0^v \frac{v'^2}{v_{te}^2} f_m(v') \frac{v'}{v_{te}} \left(\frac{v'^3}{5 v_{te}^2 v^2} - \frac{v'}{3v^2} \right) dv'$$
$$+ \int_v^\infty \frac{v'^2}{v_{te}^2} f_m(v') \frac{v'}{v_{te}} \left(\frac{v^3}{5 v_{te}^2 v'} - \frac{v}{3v'^2} \right) dv'$$