Notes on Landau damping

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In Sec. 1 of this note, resonant wave particle interactions are demonstrated by a simple test-particle simulation. Section 2 reproduces Stix's derivation of Landau damping from the view of test particles[4]. In Sec. 3, linearized Vlasov-Poison equations are solved numerically to demonstrate the "phase mix" in velocity space and the resulting Landau damping.

1 Test-particle simulation of Landau damping

Consider a longitudinal wave given by

$$\mathbf{E} = \hat{\mathbf{z}} E \cos(kz - \omega t). \tag{1}$$

The equations of motion of a test particle in the wave field are given by

$$m\frac{dv}{dt} = qE\cos(kz - \omega t), \tag{2}$$

and

$$\frac{dz}{dt} = v, \tag{3}$$

where $v \equiv \mathbf{v} \cdot \hat{\mathbf{z}}$ is the z component of the velocity of the particle.

Normalize z by the wavelength λ, t by the wave period T, v by the phase velocity v_p, i.e.,

$$\overline{z} = \frac{z}{\lambda}, \overline{t} = \frac{t}{T}, \overline{v} = \frac{v}{v_p}, \tag{4}$$

where $\lambda = 2\pi / k$, $T = 2\pi / \omega$, $v_p = \omega / k$. Using the normalized quantities, Eqs. (2) and (3) are written, respectively, as

$$\frac{d\overline{v}}{d\overline{t}} = \frac{2\pi k q E}{m\omega^2}\cos[2\pi(\overline{z} - \overline{t})], \tag{5}$$

and

$$\frac{d\overline{z}}{d\overline{t}} = \overline{v}. \tag{6}$$

The initial distribution function of particles $f_0(z,\ v)$ is taken to be uniform in space and Maxwellian in velocity,

$$f_0(z, v) = f_m(v) = \frac{1}{v_t\sqrt{2\pi}}\exp\left(-\frac{v^2}{2v_t^2}\right), \tag{7}$$

which satisfies the normalization condition $\int_{-\infty}^{\infty} f_0(v) dv = 1$ (Note that here $v_t = \sqrt{T/m}$, which is different from the usual definition $v_t = \sqrt{2T/m}$ used Sec. 3.4 of this note). In my particle simulation code (/home/yj/project_new/pic_code), 4×10^5 particles are initially loaded random in z and Maxwellian in v. Then the motion equations of every particle are followed numerically to obtain the location and velocity at later time. In the numerical code, when a particle leaves from the region $0 \leqslant \overline{x} \leqslant 1$, it is shifted by one wavelength to return to this region. This shift does not influence the force on the particle and it simulates the situation of infinite length in z direction, where when a particle leave the region $0 \leqslant \overline{x} \leqslant 1$ from the right boundary, a particle of the same velocity will enter the region from the left boundary, and vice versa.

The velocity distribution at later time is obtained by counting the number of particles in each velocity interval. Figure 1a compares the velocity distribution function at $\bar{t} = 0$ and $\bar{t} = 10$, which shows that the distribution is flatted in the resonant region $v / v_p = 1$, which suggests that the total kinetic energy of particles may be increased. Figure 2 plots the temporal evolution of the total kinetic energy of the particles, which confirms that the kinetic energy is increased by the wave. The conservation of energy tell us that the increased kinetic energy of particles must be drawn from the wave, i.e., the wave encounters damping.

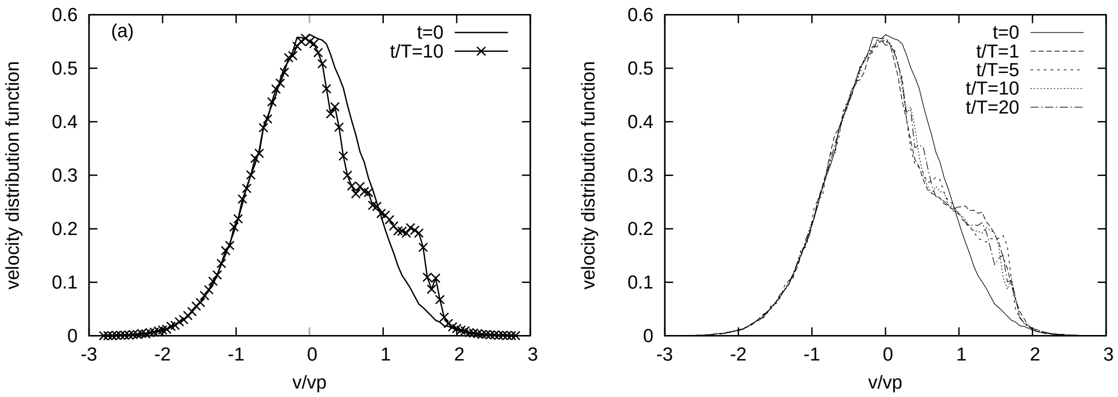

Figure 1. Comparison of the velocity distribution function (spatially averaged) at various time, which shows that the distribution is distorted in the resonant region ($v / v_p \approx 1$). Other parameters: $v_t / v_p = 1$, $2\pi kqE / m\omega^2 = 1$.

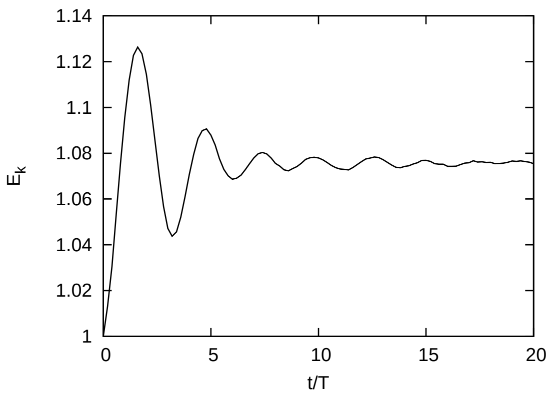

Figure 2. Temporal evolution of the total kinetic energy of the particles, where $\overline{E}_k = \sum_{i=1}^{i=N} \frac{1}{2} m v_i^2 / \left(N \frac{1}{2} m v_t^2\right)$. Other parameters: $v_t / v_p = 1$, $2\pi kqE / m\omega^2 = 1$.

Although the above simulation is performed by holding the wave amplitude constant, it takes into account all the nonlinear physics of the particle motion in the wave field. Therefore this is a nonlinear simulation.

1.1 Linear Landau damping

In the early phase of the simulation

$$t \ll \sqrt{2\pi} / \omega_b, \tag{8}$$

where ω_b is the bounce angular frequency of particles in the trough of the wave, the trapped particles effects can be neglected. This phase can be considered as the linear phase where the linear Landau damping theory (discussed later in this note) is valid. The bounce angular frequency of particles in the trough of the wave is given by[1]

$$\omega_b = \sqrt{\frac{kqE}{m}}. \tag{9}$$

Using this, the condition (8) is written as

$$\frac{t}{T} \ll \sqrt{\frac{m\omega^2}{2\pi kqE}}. \tag{10}$$

This condition reduces to $t/T \ll 1$ for the case plotted in Fig. 2, where we see that the total kinetic energy of the particles increases monotonously with time during this period. From the data in Fig. 2, the temporal change rate of the total kinetic energy can be computed, which gives $d\,\overline{E}_k/d\bar{t} = 0.08$. Next, I compare this result with those given by the analytic formula (39) (given later in this note), which is written

$$\frac{dE_k}{dt} = -\frac{\pi\omega}{|k|k}\frac{(qE)^2}{2m}\left[\frac{df(v_0)}{dv_0}\right]_{v_0=\omega/k}.$$

Using Eq. (7) and $v_t/v_p = 1$, the above expression is written

$$\frac{dE_k}{dt} = \pi\frac{(qE)^2}{2m\omega}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\right)$$

Multiplying by T and then dividing by $mv_t^2/2$, the above expression is written

$$\frac{d\,\overline{E}_k}{d\,\bar{t}} = 4\pi^2\frac{(kqE)^2}{m^2\omega^4}\frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{1}{2}\right)$$

In the simulation $2\pi kqE/m\omega^2 = 1$. Using this, expression (10) is written as

$$\frac{d\,\overline{E}_k}{d\,\bar{t}} = \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{1}{2}\right) = 0.12$$

The result given by the analytic formula is slightly different from that of the simulation (0.12 vs 0.08). Considering the various approximations used in deriving the analytic formula, the two results can be considered to be in agreement with each other.

1.2 Nonlinear Landau damping

In the phase after the linear phase, Fig. 2 indicates that the total kinetic energy of the particles oscillates with time, with the saturation level larger than the initial kinetic energy of the particles, i.e., there is nonzero net energy drawn from the wave by the particles. This is the nonlinear Landau damping, which clearly demonstrates that net energy exchange between waves and particles can be nonzero on the long time scale without including any collisional effect.

1.3 Inverse Landau damping

Choose a drift-Maxwellian distribution

$$f_0(z,v) = f_m(v) = \frac{1}{v_t\sqrt{2\pi}}\exp\left(-\frac{(v-v_b)^2}{2v_t^2}\right), \tag{11}$$

with $v_b/v_p = 2$. Then the derivative of the distribution function with respect to the velocity in the resonant region is positive. This is the case where an inverse Landau damping is expected to appear. Figure 3 compares the velocity distribution function at $\bar{t} = 0$ and $\bar{t} = 10$, which shows that the distribution is flatted in the resonant region $v/v_p = 1$. Figure 4 plots the temporal evolution of the total kinetic energy of the particles, which confirms that the kinetic energy is reduced by the wave.

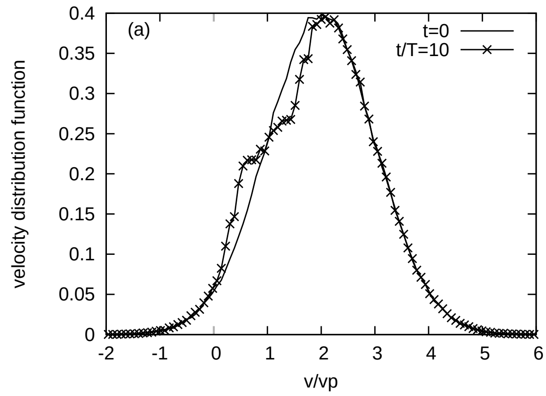

Figure 3. Comparison of the velocity distribution function (spatially averaged) at various time, which shows that the distribution is distorted in the resonant region ($v / v_p \approx 1$). Other parameters: $v_t / v_p = 1$, $2\pi k q E / m\omega^2 = 1$.

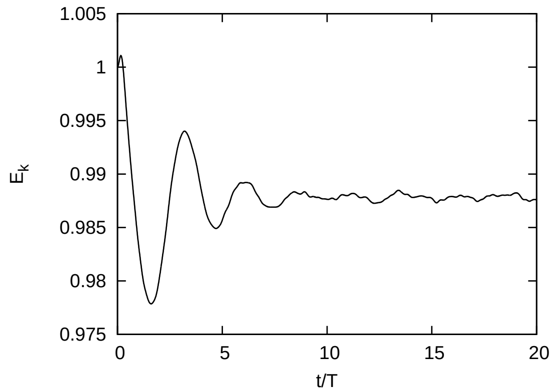

Figure 4. Temporal evolution of the total kinetic energy of the particles, where $\overline{E}_k = \sum_{i=1}^{i=N} \frac{1}{2} m v_i^2 / E_{k0}$, E_{k0} is the initial total kinetic energy. Other parameters: $v_t / v_p = 1$, $2\pi k q E / m\omega^2 = 1$.

1.4 Density fluctuation induced by the wave

Figure (5) compares the spatial distribution at $t = 0$ and $t = 10T$, which shows that the distribution become nonuniform at $t = 10T$.

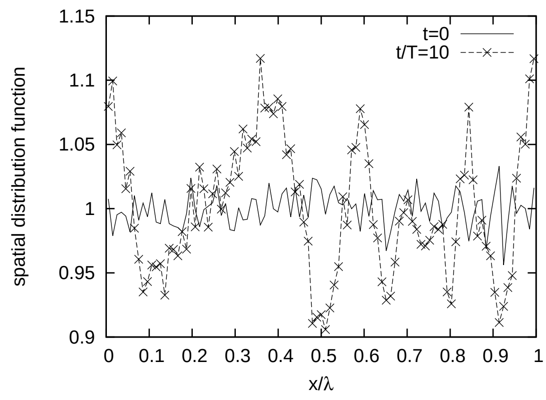

Figure 5. Comparison of the spatial distribution function at $\bar{t} = 0$ and $\bar{t} = 10$, which shows that the distribution seems to become nonuniform at $\bar{t} = 10$. Other parameters: $v_t = v_p$, $2\pi k q E / m\omega^2 = 1$.

Figure 6 is a GIF animation, which shows the time evolution of the spatial and velocity distribution (spatially averaged) of the particles. The GIF animation can be viewed only in the HTML version of this document (it does not work in the PDF version). As the animation shows, the distribution function in the resonant region ($v/v_p \approx 1$) oscillates with large amplitude at early stage, and then the amplitude becomes smaller and saturated. The spatial distribution also oscillates with large amplitude at early stage and then become much smaller and saturated. The spatial fluctuation of the density induced by the longitude wave may explain the density pump out phenomina induced by low-hybrid waves observed in many tokamaks.

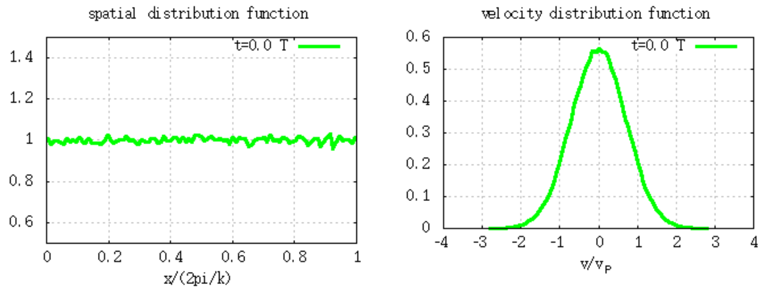

Figure 6. GIF animation of the time evolution of the spatial and velocity distribution functions during $t = [0, 20T]$, where T is the period of the longitudinal wave. Other parameters: $v_t = v_p$, $2\pi k q E / m\omega^2 = 1$. The GIF animation can be viewed only in the HTML version of this document (it does not work in the PDF version).

The above simulation is in the test-particle approximation, which means the wave is given and is not necessarily a self-consistent field. A self-consistent nonlinear simulation is given in my notes /home/yj/theory/particle_simulation.tm.

2 Analytical theory of wave particle interaction

In this section, I reproduced Stix's derivation of Landau damping from the view of test particles[4]. This method is essentially similar to the particle simulation presented in Sec. 1. The differences are that (1) we use approximate analytical method, instead of numerical one, to solve the nonlinear equation of motion of particles in wave field; (2) we focus on the particles in the resonant region $v \approx v_p$; The analytical method used here is the iterative methods often used in approximate theories[2] (Note that Stix's method is a mix of expansion and iterative methods while the method used here is a standard iterative method[2]). The contribution of the wave field to the particle motion is retained to the second order of the amplitude.

2.1 Iterative method for solving equation of motion in wave field

Consider the motion of a test particle moving in a longitudinal wave,

$$\mathbf{E} = \hat{\mathbf{z}} E \cos(kz - \omega t), \tag{12}$$

then the equation of motion is given by

$$m \frac{dv}{dt} = qE \cos(kz - \omega t), \tag{13}$$

and

$$\frac{dz}{dt} = v, \tag{14}$$

with initial condition $v(0) = v_0$ and $z(0) = z_0$, where $v \equiv \mathbf{v} \cdot \hat{\mathbf{z}}$ and $\mathbf{v}$ is the velocity of the particle. Equations (13) and (14) are nonlinear system, for which exact solutions are hard to be found. Here, we consider the amplitude of the electrical field, E, as a small perturbation, and use the iterative method[2] to solve Eqs. (13) and (14) approximately. The initial guess of the solution is obtained by setting $E = 0$, which gives

$$v^{(0)} = v_0, \tag{15}$$

and

$$z^{(0)} = z_0 + v_0 t. \tag{16}$$

Substituting this solution back into the right-hand side of Eq. (13), we obtain

$$m\frac{dv^{(1)}}{dt} = qE\cos(kz_0 + kv_0 t - \omega t), \tag{17}$$

which can be integrated over time to give

$$v^{(1)} = v_0 + \frac{qE}{m}\frac{\sin(kz_0 + \alpha t) - \sin(kz_0)}{\alpha}, \tag{18}$$

where $\alpha = kv_0 - \omega$ and use has been made of the initial condition $v^{(1)}(0) = v_0$. Substituting this solution for the velocity, Eq. (14) is written

$$\frac{dz^{(1)}}{dt} = v_0 + \frac{qE}{m}\frac{\sin(kz_0 + \alpha t) - \sin(kz_0)}{\alpha}, \tag{19}$$

which can be integrated over time, giving

$$\begin{aligned} z^{(1)} &= z_0 + v_0 t + \frac{qE}{m}\int_0^t \frac{\sin(kz_0 + \alpha t) - \sin(kz_0)}{\alpha} dt, \\ &= z_0 + v_0 t + \frac{qE}{m}\int_0^t \frac{\sin(kz_0 + \alpha t) - \sin(kz_0)}{\alpha^2} d(kz_0 + \alpha t), \\ &= z_0 + v_0 t + \frac{qE}{m}\left[\frac{-\cos(kz_0 + \alpha t) + \cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right], \end{aligned} \tag{20}$$

where use has been made of the initial condition $z^{(1)}(0) = z_0$. Substituting the solution in Eq. (20) back into the right-hand side of Eq. (13), we obtain

$$\begin{aligned} m\frac{dv^{(2)}}{dt} &= qE\cos(kz^{(1)} - \omega t) \\ &= qE\cos\left\{kz_0 + kv_0 t + k\frac{qE}{m}\left[\frac{-\cos(kz_0 + \alpha t) + \cos(kz_0)}{\alpha^2} - \frac{t\sin(kz_0)}{\alpha}\right] - \omega t\right\} \\ &= qE\cos\left\{kz_0 + \alpha t + k\frac{qE}{m}\left[\frac{-\cos(kz_0 + \alpha t) + \cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]\right\}. \end{aligned} \tag{21}$$

Since E is considered to be a small parameter, the term proportional to E can be considered to be small when compared with $kz_0 + \alpha t$. Therefore, we expand the first cosine function in the vicinity of $kz_0 + \alpha t$. Thus the above equation is written approximately as

$$m\frac{dv^{(2)}}{dt} \approx qE\ \cos(kz_0 + \alpha t) - \sin(kz_0 + \alpha t)k\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0 + \alpha t) + \cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]. \tag{22}$$

Next, calculate the time change rate of the kinetic energy of the particle, which is written as

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dt} \approx mv^{(1)}\frac{dv^{(2)}}{dt} \quad (23)$$

Using Eq. (18) for $v^{(1)}$ and Eq. (22) for $mdv^{(2)}/dt$, Eq. (23) is written

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}mv^2\right) \approx& \left[v_0 + \frac{qE}{m}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha}\right] \\ &\times \left\{qE\cos(kz_0+\alpha t) - \sin(kz_0+\alpha t)k\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]\right\} \\ \approx& \, v_0 qE\cos(kz_0+\alpha t) \\ &- v_0\sin(kz_0+\alpha t)k\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right] \\ &+ \frac{(qE)^2}{m}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha}\cos(kz_0+\alpha t), \quad (24)\end{aligned}$$

where the terms of order E^3 have been neglected (if terms of order E^3 and higher are included, then the result will correspond to nonlinear Landau damping, is this correct?). Equations (24) agrees with Eq. (8) in Chapter 8 of Stix's book[4] (however Stix's formula misses, by mistakes, the first term of the above equation).

2.2 Averaging over initial spatial location and velocity of particles

Assume the distribution function of the particles is given by $F(v_0, z_0) = f(v_0)h(z_0)$ and $h(z_0) = 1$, i.e. the distribution is uniform in space.

Consider the averaging over the initial position of particles. Define

$$\langle \ldots \rangle_{z_0} \equiv \frac{k}{2\pi}\int_0^{2\pi/k}(\ldots)h(z_0)dz_0, \quad (25)$$

which is an operator averaging over the initial position of particles in the interval of one wave length. Using this operator on both sides of Eq. (24), we obtain

$$\begin{aligned}\left\langle \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle_{z_0} =& \langle v_0 qE\cos(kz_0+\alpha t)\rangle - \left\langle \sin(kz_0+\alpha t)k v_0\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]\right\rangle + \\ &\left\langle \frac{(qE)^2}{m}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha}\cos(kz_0+\alpha t)\right\rangle \quad (26) \\ =& \, 0 - \frac{k}{2\pi}kv_0\frac{(qE)^2}{m}\int_0^{2\pi/k}\sin(kz_0+\alpha t)\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]dz_0 + \frac{k}{2\pi}\frac{(qE)^2}{m}\int_0^{2\pi/k}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha}\cos(kz_0+\alpha t)dz_0. \quad (27)\end{aligned}$$

Note that the term $v_0 qE\cos(kz_0+\alpha t)$ corresponds to the power of the electric field acting on those particles that move at a constant speed v_0. Also note this term is reduced to zero when averaged over the initial position z_0 no mater whether it is a resonant particle (i.e., $\alpha \approx 0$) or not (i.e., $\alpha \neq 0$). (This important fact is seldom mentioned in textbooks, which is one of the motivations that I wrote this note.) Changing to the new variable $x \equiv kz_0 + \alpha t$, the above equation is written as

$$\begin{aligned}
\left\langle \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle_{z_0} &= -\frac{k}{2\pi}kv_0\frac{(qE)^2}{m}\int_{\alpha t}^{2\pi+\alpha t}\sin(x)\left[\frac{-\cos(x)+\cos(x-\alpha t)}{\alpha^2}-\frac{\sin(x-\alpha t)}{\alpha}t\right]\frac{1}{k}dx + \\
&\quad \frac{k}{2\pi}\frac{(qE)^2}{m}\int_{\alpha t}^{2\pi+\alpha t}\frac{\sin(x)-\sin(x-\alpha t)}{\alpha}\cos(x)\frac{1}{k}dx \\
&= -\frac{1}{2\pi}k\,v_0\frac{(qE)^2}{m}\int_{\alpha t}^{2\pi+\alpha t}\sin(x)\left[\frac{\cos(x-\alpha t)}{\alpha^2}-\frac{\sin(x-\alpha t)}{\alpha}t\right]d\,x - \\
&\quad \frac{1}{2\pi}\frac{(qE)^2}{m}\int_{\alpha t}^{2\pi+\alpha t}\frac{\sin(x-\alpha t)}{\alpha}\cos(x)dx \\
&= -\frac{1}{2\pi}kv_0\frac{(qE)^2}{m}\left[\frac{1}{\alpha^2}\int_{\alpha t}^{2\pi+\alpha t}\sin(x)\cos(x-\alpha t)dx - \frac{t}{\alpha}\int_{-\alpha t}^{2\pi-\alpha t}\sin(x)\sin(x-\right. \\
&\quad \left.\alpha t)\,dx\right] - \frac{1}{2\pi}\frac{(qE)^2}{m}\frac{1}{\alpha}\int_{\alpha t}^{2\pi+\alpha t}\sin(x-\alpha t)\cos(x)dx \\
&= -\frac{1}{2\pi}kv_0\frac{(qE)^2}{m}\left[\frac{\pi}{\alpha^2}\sin(\alpha t)-\frac{t}{\alpha}\pi\cos(\alpha t)\right]+\frac{1}{2\pi}\frac{(qE)^2}{m}\frac{1}{\alpha}\pi\sin(\alpha t) \\
&= \frac{(qE)^2}{2m}\left[-kv_0\frac{1}{\alpha^2}\sin(\alpha t)+kv_0\frac{t}{\alpha}\cos(\alpha t)+\frac{1}{\alpha}\sin(\alpha t)\right] \\
&= \frac{(qE)^2}{2m}\left[-(\alpha+\omega)\frac{1}{\alpha^2}\sin(\alpha t)+(\alpha+\omega)\frac{t}{\alpha}\cos(\alpha t)+\frac{1}{\alpha}\sin(\alpha t)\right] \qquad (28) \\
&= \frac{(qE)^2}{2m}\left[-\frac{\omega}{\alpha^2}\sin(\alpha t)+t\cos(\alpha t)+\frac{\omega t}{\alpha}\cos(\alpha t)\right], \qquad (29)
\end{aligned}$$

which agrees with Eq. (8) in Chapter 8 of Stix's book. Next, we will average Eq. (29) over the distribution of initial velocity. Define the averaging operator in velocity space

$$\langle \ldots \rangle_{v_0} = \int_{-\infty}^{\infty}(\ldots)\,f(v_0)dv_0, \qquad (30)$$

where $f(v_0)$ is the one-dimensional distribution function, which satisfies the following normalizing condition

$$\int_{-\infty}^{\infty} f(v_0)dv_0 = 1. \qquad (31)$$

Changing to the variables $\alpha \equiv kv_0 - \omega$, equation (30) is written as

$$\langle \ldots \rangle_{v_0} = \frac{1}{|k|}\int_{-\infty}^{\infty}(\ldots)\,f\left(\frac{\alpha+\omega}{k}\right)d\alpha, \qquad (32)$$

Define

$$g(\alpha) \equiv f\left(\frac{\alpha+\omega}{k}\right). \qquad (33)$$

then equation (32) is written as

$$\langle \ldots \rangle_{v_0} = \frac{1}{|k|}\int_{-\infty}^{\infty}(\ldots)g(\alpha)d\alpha, \qquad (34)$$

Taking the average over the initial velocity, Eq. (29) is written as

$$\begin{aligned}
\left\langle\left\langle \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle_{z_0}\right\rangle_{v_0} &= \left\langle \frac{(qE)^2}{2m}\left[-\frac{\omega}{\alpha^2}\sin(\alpha t)+t\cos(\alpha t)+\frac{\omega t}{\alpha}\cos(\alpha t)\right]\right\rangle_{v_0} \\
&= \frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\left[-\frac{\omega}{\alpha^2}\sin(\alpha t)+t\cos(\alpha t)+\frac{\omega t}{\alpha}\cos(\alpha t)\right]g(\alpha)d\alpha \qquad (35)
\end{aligned}$$

It can be proved that the integration $\int_{-\infty}^{\infty} t\cos(\alpha t)g(\alpha)d\alpha$ and $\int_{-\infty}^{\infty}\frac{t}{\alpha}\cos(\alpha t)g(\alpha)d\alpha$ in the above equation approach zero rapidly for large t (refer to Sec. 2.5.3). Thus, in the sense of time asymptotic, we are left with only the integration of the first term, which is written as

$$-\frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\frac{\omega}{\alpha^2}\sin(\alpha t)g(\alpha)d\alpha. \qquad (36)$$

2.3 Resonant particles

Since there is a $1/\alpha^2$ factor in the integrand of the above integral, the important contribution to the integral must come from the vicinity of $\alpha=0$ (i.e. resonant particles). Therefore we expand $g(\alpha)$ as

$$g(\alpha)=g(0)+g'(0)\alpha+g''(0)\frac{\alpha^2}{2}+\ldots \tag{37}$$

Since $\sin(\alpha t)/\alpha^2$ is odd in α, only terms that are also odd need to be retained in the above expansion. Using these, expression (36) is written as

$$\begin{aligned}
-\frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\frac{\omega}{\alpha^2}\sin(\alpha t)g(\alpha)d\alpha &\approx -\frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\frac{\omega}{\alpha^2}\sin(\alpha t)(g'(0)\alpha)d\alpha \\
&= -\frac{1}{|k|}\frac{(qE)^2}{2m}g'(0)\omega\int_{-\infty}^{\infty}\frac{1}{\alpha}\sin(\alpha t)d\alpha \\
&= -\frac{1}{|k|}\frac{(qE)^2}{2m}g'(0)\omega\pi \\
&= -\frac{\pi\omega}{|k|k}\frac{(qE)^2}{2m}\left[\frac{df(v_0)}{dv_0}\right]_{v_0=\omega/k}.
\end{aligned} \tag{38}$$

Using these results, Eq. (35) is written as

$$\left\langle\left\langle\frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle_{z_0}\right\rangle_{v_0}\approx-\frac{\pi\omega}{|k|k}\frac{(qE)^2}{2m}\left[\frac{df(v_0)}{dv_0}\right]_{v_0=\omega/k}. \tag{39}$$

which agrees with Eq. (16) in Chapter 8 of Stix's book[4]. Equation (39) indicates that the time rate of change of the averaged kinetic energy of resonant particles is proportional to the derivative of the **initial** distribution function at the phase velocity of the wave.

2.4 Wave damping

If the longitudinal wave is an electron plasma wave, then the wave energy consists of two components, the energy of the electric field and the averaged kinetic energy of the particle oscillations,

$$W=W_E+W_p, \tag{40}$$

where W_E is the energy density of the electric field averaged in one wavelength, which is given by

$$W_E=\frac{k}{2\pi}\int_0^{2\pi/k}\frac{\varepsilon_0}{2}[E\cos(kx)]^2dx=\frac{1}{4}\varepsilon_0E^2, \tag{41}$$

W_p is the averaged kinetic energy of the particle oscillations, which, for electron plasma wave, is equal to the electric field energy W_E[3]. Using these results, equation (40) is written

$$W=\frac{1}{2}\varepsilon_0E^2. \tag{42}$$

The energy conservation requires that the kinetic energy gained by the resonant particles must come from the wave energy, i.e.,

$$\frac{dW}{dt}=\frac{\pi\omega}{|k|k}\frac{(qE)^2}{2m}\left[\frac{df(v_0)}{dv_0}\right]_{v_0=\omega/k}. \tag{43}$$

Using Eq. (42), equation (43) can be written

$$\frac{dE}{dt}=\frac{\pi\omega\omega_p^2}{2|k|k}\frac{1}{n_0}\left[\frac{df(v_0)}{dv_0}\right]_{v_0=\omega/k}E, \tag{44}$$

where $\omega_p = \sqrt{n_0 q^2 / m\varepsilon_0}$ is the electron plasma frequency. Define

$$\gamma = \frac{\pi\omega\omega_p^2}{2|k|k} \frac{1}{n_0} \left[\frac{df(v_0)}{dv_0} \right]_{v_0=\omega/k} \tag{45}$$

then Eq. (44) is written

$$\frac{dE}{dt} = \gamma E,$$

which can be integrated to give

$$E(t) = E(0)e^{\gamma t}. \tag{46}$$

The damping rate of the amplitude of the electric field given by Eq. (45) agrees the Landau damping in the weak growth rate approximation (equation (8-19) in Stix's book[4]).

2.5 Summary and discussions

In the above, we calculate the average power absorbed by a group of resonant particles moving in a longitude wave. The result [Eq. (39)] indicates that (1) if $\omega / k > 0$ and $\left[\frac{df(v_0)}{dv_0} \right]_{v_0=\omega/k} < 0$, then the power is positive, which means the particles get energy from the wave, which further means the wave are damped. (2) if $\omega / k < 0$ and $\left[\frac{df(v_0)}{dv_0} \right]_{v_0=\omega/k} > 0$, then the power is also positive, which also means the wave are damped. The two cases [(1) and (2)] can be summarized in a simple sentence: If there are more resonant particles moving slower than the wave phase velocity than those moving faster, then the wave is damped (where the resonant particles refer to the particles with $v \approx \omega / k$).

The result given above is obtained in the test particle approximation, which means the wave is given and is not necessarily a self-consistent field.

2.5.1 Lower order approximation

In Eq. (23), the time change rate of the kinetic energy of a particle (the absorbed power by the particle) is approximated by

$$\frac{d}{dt}\left(\frac{1}{2} m v^2 \right) = v^{(1)} m \frac{dv^{(2)}}{dt}, \tag{47}$$

which uses a high order approximation of the velocity $v^{(2)}$. Next, we consider lower order approximations of $d(mv^2 / 2)dt$ and check whether Landau damping can be recovered in these lower order approximations. If we approximate $d(mv^2/2)dt$ as

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} m v^2 \right) &\approx v^{(0)} m \frac{dv^{(1)}}{dt} \\ &= v_0 qE \cos(kz_0 + kv_0 t - \omega t), \end{aligned} \tag{48}$$

then it is obvious that $d(mv^2 / 2)dt$ will reduce to zero when it is averaged over initial position z_0 in one wavelength. Therefore, Landau damping is missed in this approximation. If we approximate $d(mv^2/2)dt$ as

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2} m v^2 \right) &\approx v^{(1)} m \frac{dv^{(1)}}{dt} \\ &= \left[v_0 + \frac{qE}{m} \frac{\sin(kz_0 + \alpha t) - \sin(kz_0)}{\alpha} \right] qE \cos(kz_0 + kv_0 t - \omega t), \end{aligned} \tag{49}$$

then, according to the derivation given in the above section, we have

$$\left\langle \frac{d}{dt}\left(\frac{1}{2} m v^2 \right) \right\rangle_{z_0} = \frac{(qE)^2}{2m} \frac{1}{\alpha} \sin(\alpha t) \tag{50}$$

and

$$\left\langle \left\langle \frac{d}{dt}\left(\frac{1}{2} m v^2 \right) \right\rangle_{z_0} \right\rangle_{v_0} = \frac{1}{|k|} \frac{(qE)^2}{2m} \int_{-\infty}^{\infty} \frac{1}{\alpha} \sin(\alpha t) g(\alpha) d\alpha. \tag{51}$$

The right-hand side of Eq. (50) do not approach zero for large t (I have verified this for the case that $g(\alpha) = 1/(1+\alpha^2)$). Since there is a $1/\alpha$ factor in the integrand of the above integral, the important contribution to the integral must come from the vicinity of $\alpha = 0$. Therefore we expand $g(\alpha)$ as

$$g(\alpha) = g(0) + g'(0)\alpha + g''(0)\frac{\alpha^2}{2} + \ldots \tag{52}$$

Since $\sin(\alpha t)/\alpha$ is even in α, only terms that are also even need to be retained in the above expansion. Using these, Eq. (50) is written as

$$\begin{aligned}\left\langle\left\langle \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\right\rangle_{z_0}\right\rangle_{v_0} &= \frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\frac{1}{\alpha}\sin(\alpha t)g(\alpha)d\alpha \\ &\approx \frac{1}{|k|}\frac{(qE)^2}{2m}\int_{-\infty}^{\infty}\frac{1}{\alpha}\sin(\alpha t)g(0)d\alpha \\ &= \frac{\pi}{|k|}\frac{(qE)^2}{2m}g(0),\end{aligned} \tag{53}$$

which is obviously not Landau damping. Then what is this contribution? The answer is that it is an "error term" often encountered in the iterative method. In the above section, we saw that the term $\sin(\alpha t)/\alpha$ was canceled when we go higher order approximation (refer to the derivation between Eqs. (28) and (29)). The iterative method has the undesirable feature that in the early iteration it gives erroneous values to the higher order terms. One can only check that a term is correct by making one more iteration, which of course is usually convincing but no rigorous proof. Therefore, strictly speaking, the result given here does not prove the existence of Landau damping, but only suggests that the Landau damping is very likely to exist.

2.5.2 Using power

The power on a particle is the velocity multiplied by the force, i.e.,

$$\begin{aligned}P &= Fv \\ &= Eq\cos(kz-\omega t)v.\end{aligned} \tag{54}$$

Different order approximations of v and z can be used in Eq. (54) to evaluate the power. If the approximations $v \approx v^{(0)} = v_0$ and $z \approx z^{(0)} = z_0 + v_0 t$ are used, Eq. (54) is written as

$$P \approx Eq\cos(kz_0 + \alpha t)v_0, \tag{55}$$

which is obviously zero when it is averaged over initial position z_0 in one wavelength. If the approximations and $v \approx v^{(1)}$ and $z \approx z^{(1)}$ are used, Eq. (54) is written as

$$\begin{aligned}P \approx\ & \left\{qE\cos(kz_0+\alpha t) - \sin(kz_0+\alpha t)k\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]\right\}\left[v_0 + \right. \\ & \left.\frac{qE}{m}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha}\right] \\ \approx\ & Eq\cos(kz_0+\alpha t)v_0 + Eq\cos(kz_0+\alpha t)\frac{qE}{m}\frac{\sin(kz_0+\alpha t)-\sin(kz_0)}{\alpha} - \sin(kz_0 + \\ & \alpha t)k\frac{(qE)^2}{m}\left[\frac{-\cos(kz_0+\alpha t)+\cos(kz_0)}{\alpha^2} - \frac{\sin(kz_0)}{\alpha}t\right]v_0,\end{aligned} \tag{56}$$

where, the term proportional to E^3 has been neglected. Equation (56) is identical with Eq. (24). Therefore the derivation after this point is the same as given in Sec. 2.2.

2.5.3 Time asymptotic behavior of integral $\int_{-\infty}^{\infty} t\cos(\alpha t)g(\alpha)d\alpha$

Consider the concrete case that $g(\alpha) = 1/(1+\alpha^2)$, the integration can be performed analytically (by using Wolfram Mathematica), which gives

$$\begin{aligned}\int_{-\infty}^{\infty} t\cos(\alpha t)g(\alpha)d\alpha &= \int_{-\infty}^{\infty}\cos(x)g(\frac{x}{t})dx \\ &= \pi\frac{e^{-t}}{t},\end{aligned} \tag{57}$$

which rapid approaches zero for large t.

3 Self-consistent-field linear simulation of Landau damping

3.1 One-dimensional Vlasov-Poisson equations

The test-particle simulation of Landau damping is given is Sec. 1. In this section, we consider the self-consistent simulation of the linear Landau damping. The Vlasov equation is written

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f=0, \tag{58}$$

where f is the electron distribution function, q and m are the charge and mass of electrons, respectively. The linearized version of the above equation is written

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1+\frac{q}{m}(\mathbf{E}_0+\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_v f_1=-\frac{q}{m}(\mathbf{E}_1+\mathbf{v}\times\mathbf{B}_1)\cdot\nabla_v f_0. \tag{59}$$

We consider the case of $\mathbf{B}_0=0$ and $\mathbf{E}_0=0$. Further consider only the electrostatic case, i.e. $\partial\mathbf{B}_1/\partial t=0$, i.e., there is not magnetic fluctuation. Then the linearized Vlasov equation (59) is written

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla f_1=-\frac{q}{m}\mathbf{E}_1\cdot\nabla_v f_0. \tag{60}$$

Consider the one-dimensional case where f_1, and $\mathbf{E}_1$ are both independent of x and y coordinates and $\mathbf{E}_1=E_1\hat{\mathbf{z}}$, then the above equation is written

$$\frac{\partial f_1}{\partial t}+v_z\frac{\partial f_1}{\partial z}=-\frac{q}{m}E_1\frac{\partial f_0}{\partial v_z}. \tag{61}$$

Integrating both sides of Eq. (61) over v_x and v_y, we obtain

$$\frac{\partial F_1}{\partial t}+v_z\frac{\partial F_1}{\partial z}=-\frac{q}{m}E_1\frac{\partial F_0}{\partial v_z}, \tag{62}$$

where

$$F_0=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_0\,dv_x dv_y, \tag{63}$$

and

$$F_1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_1\,dv_x dv_y, \tag{64}$$

are the reduced distribution functions. Poisson's equation is written

$$\frac{\partial E}{\partial z}=\frac{1}{\varepsilon_0}\left(n_i q_i+q\int_{-\infty}^{\infty}f d^3v\right), \tag{65}$$

where n_i and q_i are the number density and charge of ions, respectively. In equilibrium the number density of electrons and ions are equal to each other. Assuming the number density of the massive ions remain unchanged, Poisson's equation for the perturbed quantities is written

$$\frac{\partial E_1}{\partial z}=\frac{1}{\varepsilon_0}q\int_{-\infty}^{\infty}f_1\,d^3v, \tag{66}$$

In terms of the reduced distribution function, equation (66) is written

$$\frac{\partial E_1}{\partial z}=\frac{1}{\varepsilon_0}q\int_{-\infty}^{\infty}F_1\,dv_z. \tag{67}$$

Equations (62) and (67) governs the time evolution of F_1 and E_1. Consider the case that F_0 is spatially uniform, then all the coefficients of Eqs. (62) and (67) are independent of the spatial coordinate, which indicates that different spatial Fourier harmonics are decoupled. Performing Fourier transform about z on both sides of Eqs. (62) and (67), we obtain

$$\frac{\partial \hat{F}_1}{\partial t}+ikv_z\hat{F}_1=-\frac{q}{m}\hat{E}_1\frac{\partial F_0}{\partial v_z}, \tag{68}$$

$$ik\hat{E}_1=\frac{1}{\varepsilon_0}q\int_{-\infty}^{\infty}\hat{F}_1\,dv_z, \tag{69}$$

where $\hat{F}_1$ and $\hat{E}_1$ are the spatial Fourier transform of F_1 and E_1, respectively. Using Eq. (69) in Eq. (68) yields

$$\frac{\partial \hat{F}_1}{\partial t} = -ikv_z \hat{F}_1 - \frac{q^2}{\varepsilon_0 m}\frac{\partial F_0}{\partial v_z}\frac{1}{ik}\int_{-\infty}^{\infty}\hat{F}_1\, dv_z \tag{70}$$

Given an equilibrium distribution function $F_0(v_z)$ and an initial perturbation $\hat{F}_1(t=0, v_z)$, equation (70) can be solved analytically by using the Laplace transformation. Here, to avoid the fancy mathematics involved in the Laplace transformation, I solve Eq. (70) by a direct numerical method.

3.2 Normalization

The reduced equilibrium distribution function F_0 satisfies the normalization condition $\int_{-\infty}^{\infty} F_0(v_z) dv_z = n_0$, where n_0 the number density of electrons. Denote the thermal velocity of the equilibrium distribution by v_t. Then Eq. (70) can be written

$$\frac{\partial \hat{F}_1}{\partial t} = -ikv_z \hat{F}_1 - \frac{q^2 n_0}{\varepsilon_0 m}\frac{1}{ikv_t}\left(\frac{v_t}{n_0}\frac{\partial F_0}{\partial v_z}\right)\int_{-\infty}^{\infty}\hat{F}_1\, dv_z, \tag{71}$$

which can be further written

$$\frac{\partial \hat{F}_1}{\partial t} = -ikv_t \overline{v}_z \hat{F}_1 - \frac{\omega_p^2}{ikv_t}\left(\frac{v_t^2}{n_0}\frac{\partial F_0}{\partial v_z}\right)\int_{-\infty}^{\infty}\hat{F}_1\, d\overline{v}_z, \tag{72}$$

where $\overline{v}_z = v_z / v_t$, $\omega_p = \sqrt{n_0 q^2 / (\varepsilon_0 m)}$ is the electron plasma frequency. Define $\overline{t} = t\omega_p$, then Eq. (72) is written

$$\frac{\partial \hat{F}_1}{\partial \overline{t}} = -\frac{ikv_t}{\omega_p}\overline{v}_z \hat{F}_1 - \frac{\omega_p}{ikv_t}\left(\frac{v_t^2}{n_0}\frac{\partial F_0}{\partial v_z}\right)\int_{-\infty}^{\infty}\hat{F}_1\, d\overline{v}_z, \tag{73}$$

Define $a = ikv_t/\omega_p$, then the above equation is written

$$\frac{\partial \hat{F}_1}{\partial \overline{t}} = -a\overline{v}_z \hat{F}_1 - \frac{1}{a}\left(\frac{v_t^2}{n_0}\frac{\partial F_0}{\partial v_z}\right)\int_{-\infty}^{\infty}\hat{F}_1\, d\overline{v}_z, \tag{74}$$

3.3 Phase mixing and linear Landau damping

Consider the case that the equilibrium distribution F_0 is Maxwellian in velocity space:

$$F_0(v_z) = \frac{n_0}{v_t\sqrt{\pi}}\exp\left(-\frac{v_z^2}{v_t^2}\right), \tag{75}$$

which satisfies the normalization condition $\int_{-\infty}^{\infty} F_0(v_z) dv_z = n_0$. The derivative of F_0 with respect to v_z is written

$$\frac{\partial F_0}{\partial v_z} = \frac{n_0}{v_t\sqrt{\pi}}\exp\left(-\frac{v_z^2}{v_t^2}\right)\left(-\frac{2v_z}{v_t^2}\right). \tag{76}$$

Using this, Eq. (74) is written

$$\frac{\partial \hat{F}_1}{\partial \overline{t}} = -a\,\overline{v}_z \hat{F}_1 - \frac{1}{a}\frac{1}{\sqrt{\pi}}\exp(-\overline{v}_z^2)(-2\overline{v}_z)\int_{-\infty}^{\infty}\hat{F}_1\, d\overline{v}_z. \tag{77}$$

Take the initial condition of $\hat{F}_1$ to be

$$\hat{F}_1(\overline{t}=0, \overline{v}_z) = \frac{1}{\sqrt{\pi}}\exp(-\overline{v}_z^2) + \frac{i}{\sqrt{\pi}}\exp(-\overline{v}_z^2), \tag{78}$$

which is a Maxwellian distribution that satisfies the normalization $\int_{-\infty}^{\infty}\hat{F}_1(\overline{v}_z) d\,\overline{v}_z = 1 + i$. Equation (77) with the initial condition Eq. (78) was solved numerically to obtain the time evolution of $\hat{F}_1$ (the code is in /home/yj/project/landau_damping/). Figure 7 compares the velocity distribution function at $\overline{t} = 0$ with that at $\overline{t} = 40$, which shows that the distribution function develops fine structures in velocity space.

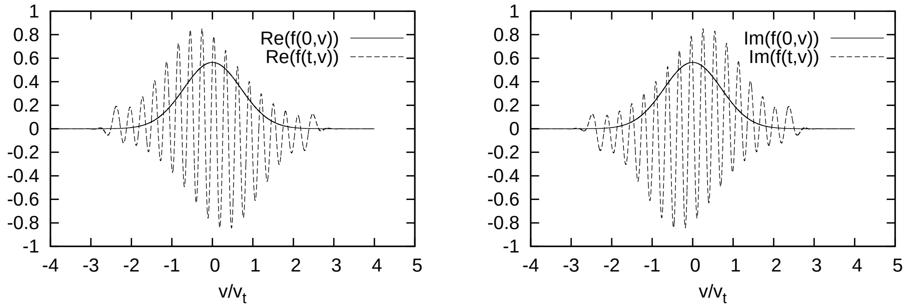

Figure 7. Comparison of $\hat{F}_1$ at $t=0$ and $t\omega_p=40$. (a) real part; (b) imaginary part. Other parameters $kv_t/\omega_p=0.5$.

It is ready to realize that the fine velocity space structures are partially due to the first term on the right-hand side of Eq. (77). When only this term is retained, Eq. (77) is written

$$\frac{\partial \hat{F}_1}{\partial \bar{t}} = -i\left(\frac{kv_t}{\omega_p}\right)\bar{v}_z \hat{F}_1, \tag{79}$$

which has the dispersion relation

$$\omega = \left(\frac{kv_t}{\omega_p}\right)\bar{v}_z, \tag{80}$$

which indicates that Eq. (79) has different eigenfrequencies for different points in velocity space. Thus, an initially rather smooth velocity distribution function will become not so smooth after some time due to the distribution function oscillate with different frequencies at different velocity points. This is the so-called "phase mixing". It is obvious that, after some time, the phase mixing will make the velocity distribution function $\hat{F}_1(v_z)$ rather messy, which poses a great challenge to the numerical resolution of $\hat{F}_1(v_z)$. Given a fixed velocity grids, the numerical results will become inaccurate when the grids is not fine enough to resolve the fine structures of velocity distribution.

Note that the electric field is related to the integration of F_1, i.e.

$$\hat{E}_1 = \frac{1}{ik}\frac{q}{\varepsilon_0}\int_{-\infty}^{\infty}\hat{F}_1\, dv_z \tag{81}$$

Then it is fairly obvious that the phase mixing have the possibility of reducing the magnitude of the perturbed electric field. Figure 8 plot the time evolution of $\hat{E}_1$ (the factor $q/(k\varepsilon_0)$ in Eq. (81) is removed), which shows that electric field oscillates with the amplitude decreasing with time. This confirms that the phase mixing reduces the magnitude of the electric field.

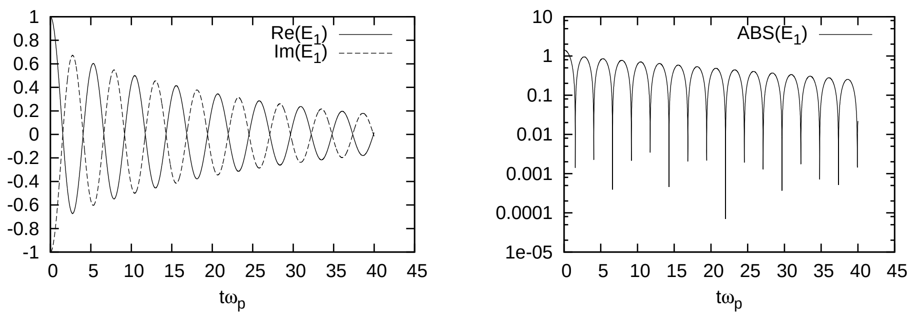

Figure 8. (a) time evolution of real and imaginary parts of $\hat{E}_1$. (b) time evolution of $|\hat{E}_1|$ in logarithm scale. $kv_t/\omega_p=0.5$.

3.4 Comparison with linear analytical theory

Next, we compare the numerical results with the electron plasma wave dispersion relation, which is given by[1]

$$1+2\left(\frac{\omega_p}{kv_t}\right)^2[1+\zeta Z(\zeta)]=0, \qquad (82)$$

where $\zeta=\omega/kv_t$, $v_t=\sqrt{2T_e/m_e}$, and

$$Z(\zeta)=\frac{1}{\sqrt{\pi}}\int_C\frac{e^{-z^2}}{z-\zeta}dz, \qquad (83)$$

is the plasma dispersion function. The plasma dispersion function can be written as

$$Z(\zeta)=i\sqrt{\pi}\exp(-\zeta^2)\mathrm{erfc}(-i\zeta), \qquad (84)$$

where erfc is the complementary error function of imaginary argument, which is implemented in Wolfram Mathematica. By using FindRoot function of Wolfram Mathematica, the equation (82) can be easily solved numerically to find the root. For the parameter used in the simulation $kv_t/\omega_p=0.5$, Findroot gives $\zeta=2.4508-i0.0725$. From this, we obtain $\omega/\omega_p=\zeta kv_t/\omega_p=1.2254-i0.0362$.

The oscillation frequency of the electric field can be estimated by counting the peaks in Fig. 8, from which we obtain $\omega_r/\omega_p=1.226$, which agrees the theoretic value 1.2254 given above. Figure 8 shows that the amplitude of the electric filed decreases exponentially with time. Figure 9 compares the theoretic growth rate with the simulation results, which also shows good agreement with each other.

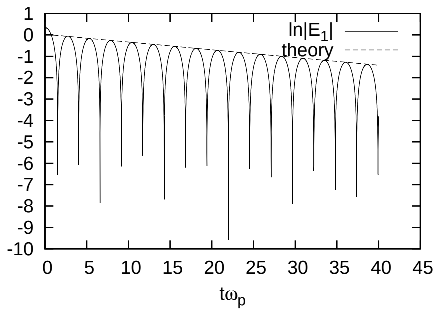

Figure 9. Comparison of the damping rate given by Eq. (82) ($\gamma/\omega_p=-0.0362$) with the simulation results for the parameter $kv_t/\omega_p=0.5$.

In the weak growth rate approximation, the real frequency of electron plasma wave is given by

$$\omega_r^2=\omega_p^2+\frac{3}{2}k^2v_t^2, \qquad (85)$$

For the numerical case given here, $kv_t/\omega_p=0.5$. Using this in Eq. (85), we obtain $\omega_r/\omega_p=1.173$, which roughly agrees with the exact value $\omega_r/\omega_p=1.2254$.

In the weak growth rate approximation, the growth rate is given by Eq. (45), i.e.,

$$\gamma=\frac{\pi\omega\omega_p^2}{2|k|k}\frac{1}{n_0}\left[\frac{df_0(v)}{dv}\right]_{v=\omega/k}, \qquad (86)$$

Using this, we obtain

$$\begin{aligned}\gamma &= \frac{\pi\omega\omega_p^2}{2k|k|}\frac{1}{n_0}\left[\frac{n_0}{v_t\sqrt{\pi}}\exp\left(-\frac{v^2}{v_t^2}\right)\left(-\frac{2v}{v_t^2}\right)\right]_{v=\omega/k}\\ &= \frac{\pi\omega\omega_p^2}{2k^2}\left[\frac{1}{v_t\sqrt{\pi}}\exp\left(-\frac{\omega^2}{k^2v_t^2}\right)\left(-\frac{2\omega}{kv_t^2}\right)\right]. \qquad (87)\end{aligned}$$

From this, we obtain

$$\frac{\gamma}{\omega_p} = \sqrt{\pi}\frac{\omega_p}{kv_t}\left[\exp\left(-\frac{\omega_r^2/\omega_p^2}{k^2v_t^2/\omega_p^2}\right)\left(-\frac{\omega_r^2/\omega_p^2}{k^2v_t^2/\omega_p^2}\right)\right]. \tag{88}$$

Using $\omega/\omega_p = 1.225$ in Eq. (88), we obtain $\gamma/\omega_p = -0.0524$, which roughly agrees with the exact value $\gamma/\omega_p = -0.0362$ obtained above. Note that if we used $\omega \approx \omega_p$, instead of the exact frequency $\omega = 1.225\omega_p$, then Eq. (88) would give $\gamma/\omega_p = -0.259$, which is almost one order larger than the exact value $\gamma/\omega_p = -0.036$. This highlights the inaccuracy of the approximate formula we encounter in textbooks[1], where $\omega = \omega_p$ is used to estimate the damping rate.

Figure 10 compares the exact numerical solution of the dispersion relation (82) with the approximate solution given by Eqs. (85) and (88). The results indicate that the approximate growth rate give by Eq. (88) is much lower than than the exact value for $kv_t/\omega_{pe} > 1$.

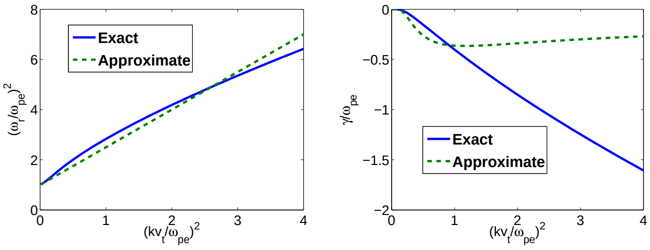

Figure 10. Comparision of the exact numerical solution of the dispersion relation (82) with the approximate solution given by Eqs. (85) and (88). Numerical calculation uses the Matlab implementation of the error function of imaginary argument by Marcel Leutenegger (January 2008). The root of the dispersion equation is found by using "fsolve" of Matlab.

3.5 On the consrvation of particle number

In the above, we see that the integration $|\int_{-\infty}^{\infty}\hat{F}_1 dv_z|$ decreases with time, which seems to be inconsistent with the conservation of particle number. Note the spatial dependence of the perturbed distribution function is e^{ikz}, i.e., the perturbed distribution function is given by $\hat{F}_1 e^{ikz}$, the real part of which corresponds to the physical distribution function, i.e., $F_1(t, v_z, z) = A(t, v)\cos(kz + \alpha)$, where $A = |\hat{F}_1|$ and α is the angle of $\hat{F}_1$ on the complex plane. The particle number for the distribution function F_1 in a region of the wave length $2\pi/k$ is given by

$$\begin{aligned} N &= \int_0^{2\pi/k}\int_{-\infty}^{\infty} F_1 dv_z dz \\ &= \int_0^{2\pi/k} dz \int_{-\infty}^{\infty} A(t,v)\cos(kz+a) dv_z \\ &= \int_{-\infty}^{\infty} A(t,v)\left(\int_0^{2\pi/k} dz\cos(kz+\alpha)\right) dv_z \end{aligned} \tag{89}$$

Note that the term in the parenthesises is zero. Therefore the above expression is always zero no matter what value the integration $\int_{-\infty}^{\infty}\hat{F}_1 dv_z$ is . Thus the number of particles is conserved in this case.

4 Two-stream instability

In Sec. 3, the equilibrium velocity distribution of electrons is chosen to be Maxwellian, where we see that perturbations are damped, which is the well-known Landau damping. In this section, we investigate a case where perturbation grows, instead of being damped. Consider an equilibrium distribution function consisting of two counter-propagating Maxwellian beams of mean speed v_b and thermal spread v_t, i.e.,

$$F_0(v_z) = \frac{n_0}{2}\left[\frac{1}{v_t\sqrt{\pi}}\exp\left(-\frac{(v_z - v_b)^2}{v_t^2}\right) + \frac{1}{v_t\sqrt{\pi}}\exp\left(-\frac{(v_z + v_b)^2}{v_t^2}\right)\right]. \tag{90}$$

Then $\partial F_0/\partial v_z$ is written

$$\frac{\partial F_0}{\partial v_z} = \frac{n_0}{2}\left[\frac{1}{v_t\sqrt{\pi}}\exp\left(-\frac{(v_z-v_b)^2}{v_t^2}\right)\left(-\frac{2\,(v_z-v_b)}{v_t^2}\right) + \frac{1}{v_t\sqrt{\pi}}\exp\left(-\frac{(v_z+v_b)^2}{v_t^2}\right)\left(-\frac{2\,(v_z+v_b)}{v_t^2}\right)\right] \tag{91}$$

Using the same code discussed in Sec. 3, I solve Equation (74) with $\partial F_0/\partial v_z$ given by Eq. (91) and the initial perturbation given by Eq. (77). Figure 11 plots the equilibrium distribution function with $v_b/v_t=4$.

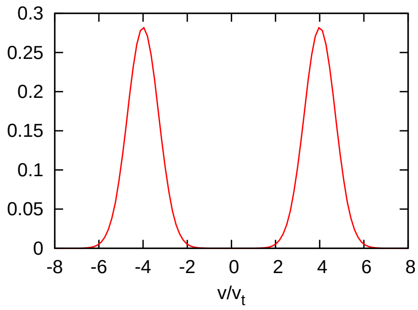

Figure 11. Equilibrium distribution function given by Eq. (90) with $v_b/v_t=4$.

Figure 12 plots the time evolution of the perturbed electric field, which shows the the electric field grows exponentially in time and thus corresponds to an instability. This instability is called two-stream instability since it happens in the system with two opposite electron beams.

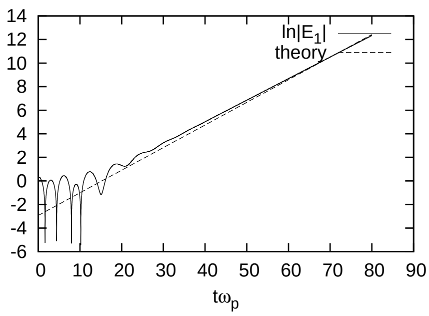

Figure 12. Comparison of the simulation results with analytical growth rate given by Eq. (92). The parameters are $kv_t/\omega_p=0.05$ and $v_b/v_t=4$.

In Fig. 12, the simulation results are also compared with the analytical results in the cold beam approximation ($v_t\rightarrow 0$), which is given by equation (8.1.35) in Gurnett's book[1], i.e.,

$$\frac{\gamma}{\omega_p} = \sqrt{\sqrt{1+4\frac{k^2v_b^2}{\omega_p^2}} - \left(1+\frac{k^2v_b^2}{\omega_p^2}\right)}, \tag{92}$$

To make the simulation result able to be compared with the results in the cold beam approximation, the thermal velocity of the beam has been chosen to be a small number $kv_t/\omega_p=0.05$. The results in Fig. 12 shows that the simulation results agree with the analytical results. The small discrepancy can be attributed to that equation (92) was derived by assuming the electron distribution function is a Dirac δ function while in the simulation, the distribution function is a Maxwellian distribution with small a thermal spread ($kv_t/\omega_p = 0.05$). Also note that in this case, the approximate phase velocity of the electron plasma wave is $v_p=\omega_p/k=20v_t$ and the beam velocity $v_b = 4v_t$. Thus, the distribution function is very small at the phase velocity. Therefore the Landau damping in this case is neglectably small. In fact, equation (92) was derived in the fluid mode in which the Landau damping is not included.

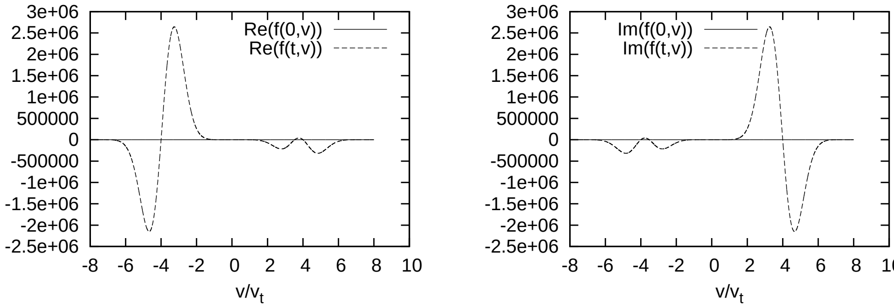

Figure 13. Real part (a) and imaginary part (b) of the perturbed distribution function $\hat{F}_1$ at $t\omega_p = 80$. The parameters are $kv_t/\omega_p = 0.05$ and $v_b/v_t = 4$.

5 tmp——to be deleted

————As is given in the wikipedia, the ponderomotive force is a nonlinear force that a charged particle experiences in an inhomogeneous oscillating electromagnetic field. The ponderomotive force $\mathbf{F}_p$ is expressed by

$$\mathbf{F}_p = -\frac{e^2}{4m\omega^2}\nabla E^2, \tag{93}$$

where e is the electrical charge of the particle, m is the mass of the particle, E is the amplitude of the inhomogeneous oscillating electric field (at low enough amplitudes the magnetic field exerts very little force), ω is the angular frequency of oscillation of the field. to be continued—

$$\varepsilon(k,\omega) = 1 + \frac{e^2}{\varepsilon_0 m_e k}\int_C \frac{\partial f_0/\partial v}{\omega - kv} dv = 0$$

$$1 - 2\frac{e^2}{\varepsilon_0 m_e k}\frac{n_0}{v_t\sqrt{\pi}}\frac{1}{kv_t}\int_C \frac{\exp(-t^2)}{\zeta - t} t\,dt = 0$$

$$Z(\zeta) = 2ie^{-\zeta^2}\int_{-\infty}^{i\zeta} e^{-t^2}dt. \tag{94}$$

$$Z(\zeta) = i\sqrt{\pi}w(\zeta), \tag{95}$$

where $w(\zeta)$ is Faddeeva's function, which is defined by

$$w(\zeta) = \exp(-\zeta^2)\mathrm{erfc}(-i\zeta). \tag{96}$$

$$\begin{aligned} Z(\zeta) &= i\sqrt{\pi}\exp(-\zeta^2)\mathrm{erfc}(-i\zeta) \\ &= i\sqrt{\pi}\exp(-\zeta^2)\frac{2}{\sqrt{\pi}}\int_{-i\zeta}^{\infty} e^{-t^2}dt \\ &= 2i\exp(-\zeta^2)\int_{-\infty}^{i\zeta} e^{-t^2}dt \end{aligned} \tag{97}$$

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