Notes on linear ideal MHD theory of tokamak plasmas

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Abstract

This note discusses the linear ideal magnetohydrodynamic (MHD) theory of tokamak plasmas. This note also serves as a document for the GTAW code (General Tokamak Alfvén Waves code), which is a Fortran code calculating Alfven eigenmodes in realistic tokamak geometry.

1 MHD equations

The time evolution of the fluid velocity $\mathbf{u}$ is governed by the momentum equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \rho_q\mathbf{E} - \nabla p + \mathbf{J}\times\mathbf{B}, \tag{1}$$

where ρ, ρ_q, p, $\mathbf{J}$, $\mathbf{E}$, and $\mathbf{B}$ are mass density, charge density, thermal pressure, current density, electric field, and magnetic field, respectively. The time evolution of the mass density ρ is governed by the mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{2}$$

The time evolution equation for pressure p is given by the equation of state

$$\frac{d}{dt}(p\rho^{-\gamma}) = 0, \tag{3}$$

where γ is the ratio of specific heats. The time evolution of $\mathbf{B}$ is given by Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}. \tag{4}$$

The current density $\mathbf{J}$ can be considered as a derived quantity, which is defined through Ampere's law (the displacement current being ignored)

$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B} \tag{5}$$

The electric field $\mathbf{E}$ is considered to be a derived quantity, which is defined through Ohm's law

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B} + \eta\mathbf{J}. \tag{6}$$

The charge density ρ_q can be considered to be a derived quantity, which is defined through Poisson's equation,

$$\rho_q = \varepsilon_0\nabla\cdot\mathbf{E}. \tag{7}$$

The above equations constitute a closed set of equations for the time evolution of four quantities, namely, $\mathbf{B}$, $\mathbf{u}$, ρ, and p (the electric field $\mathbf{E}$, current density $\mathbf{J}$, and charge density ρ_q are eliminated by using Eqs. (5), (6), and (6)). In addition, there is an equation governing the spatial structure of the magnetic field, namely

$$\nabla\cdot\mathbf{B} = 0. \tag{8}$$

In summary, the MHD equations can be categorized into three groups of equations, namely[1],

1. Evolution equations for base quantities $\mathbf{B}$, $\mathbf{u}$, ρ, and p:
   $\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times(-\mathbf{u}\times\mathbf{B} + \eta(\nabla\times\mathbf{B})/\mu_0) = \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}/\mu_0$
   $\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \rho_q\mathbf{E} - \nabla p + (\nabla\times\mathbf{B})/\mu_0\times\mathbf{B}$
   $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$
   $\frac{d}{dt}(p\rho^{-\gamma}) = 0,$
2. Equation of constraint: $\nabla\cdot\mathbf{B} = 0$.
3. Definitions: (i.e., they are considered to be derived quantities.)
   $\mathbf{E} = \eta(\nabla\times\mathbf{B})/\mu_0 - \mathbf{u}\times\mathbf{B},$
   $\mathbf{J} = (\nabla\times\mathbf{B})/\mu_0,$
   $\rho_q = \varepsilon_0\nabla\cdot\mathbf{E}.$

The electrical field term $\rho_q\mathbf{E}$ in the momentum equation (1) is usually neglected because this term is usually much smaller than other terms for low-frequency phenomena in tokamak plasmas.

1.1 Self-consistency check

It is well known that the divergence of Faraday's law (4) is written

$$\frac{\partial\nabla\cdot\mathbf{B}}{\partial t} = -\nabla\cdot(\nabla\times\mathbf{E}) = 0, \tag{9}$$

which implies that $\nabla\cdot\mathbf{B}=0$ will hold in later time if it is satisfied at the initial time.

Because the displacement current is neglected in Ampere's law, the divergence of Ampere's law is written

$$\nabla\cdot\mathbf{J}=0. \tag{10}$$

On the other hand, the charge density is defined through Poisson's equation, Eq. (7), i.e.,

$$\begin{aligned}\rho_q &\equiv \varepsilon_0\nabla\cdot\mathbf{E}\\ &= \varepsilon_0\nabla\cdot(\eta\mathbf{J}-\mathbf{u}\times\mathbf{B}).\end{aligned} \tag{11}$$

which indicates that the charge density ρ_q is usually time dependent, i.e., $\partial\rho_q/\partial t\neq 0$. Therefore the charge conservation is not guaranteed in this framework. This inconsistency is obviously due to the fact that we neglect the displacement current $\partial\mathbf{E}/\partial t$ in Ampere's law. Since, for low frequency phenomena, the displacement current $\partial\mathbf{E}/\partial t$ term is usually much smaller than the the conducting current $\mathbf{J}$, neglecting the displacement current term induces only small errors in calculating $\mathbf{J}$ by using Eq. (5).

1.2 Generalized Ohm's law

Referece [2] gives a clear derivation of the generalized Ohm's law, which takes the following form

$$\mathbf{E}+\mathbf{u}\times\mathbf{B}-\eta\mathbf{J} = \frac{1}{en}\mathbf{J}\times\mathbf{B} - \frac{1}{en}\nabla\cdot\mathbf{P}_e + \frac{m_e}{ne^2}\left[\frac{\partial\mathbf{J}}{\partial t}+\nabla\cdot(\mathbf{J}\mathbf{u}+\mathbf{u}\mathbf{J})\right], \tag{12}$$

where the first term on the right-hand side is called the "Hall term", the second term is the electron pressure term, and the third term is called the "electron inertia term" since it is proportional to the mass of electrons.

Note that both $\mathbf{u}$ and $\mathbf{J}$ are the first-order moments, with $\mathbf{u}$ being the (weighted) sum of the first-order moment of electrons and ions while $\mathbf{J}$ being the difference between them. The generalized Ohm's law is actually the difference between the electrons and ions first-order moment equations. The generalized Ohm's law is an equation that governs the time evolution of $\mathbf{J}$. Also note that Ampere's law, with the displacement current retained, is an equation governing the time evolution of $\mathbf{E}$. However, in the approximation of the resistive MHD, the time derivative terms $\partial\mathbf{E}/\partial t$ and $\partial\mathbf{J}/\partial t$ are ignored in Ampere's law and Ohm's law, respectively. In this approximation, Ohm's law is directly solved to determine $\mathbf{E}$ and Ampere's law is directly solved to determine $\mathbf{J}$.

1.3 Eliminating mass density ρ from equation of state

The equation of state (3) involves three physical quantities, namely ρ, p, and $\mathbf{u}$. It turns out that the continuity equation can be used in the equation to eliminate ρ. The equation of state

$$\frac{d}{dt}(p\rho^{-\gamma})=0, \tag{13}$$

can be written as

$$\frac{dp}{dt}\rho^{-\gamma}-\gamma\rho^{-(\gamma+1)}p\frac{d\rho}{dt}=0, \tag{14}$$

which simplifies to

$$\frac{dp}{dt}=\gamma\frac{p}{\rho}\frac{d\rho}{dt} \tag{15}$$

Expand the total derivative, giving

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p=\gamma\frac{p}{\rho}\left(\frac{\partial\rho}{\partial t}+\mathbf{u}\cdot\nabla\rho\right) \tag{16}$$

Using the mass continuity equation to eliminate ρ in the above equation gives

$$\frac{\partial p}{\partial t}=-\gamma p\nabla\cdot\mathbf{u}-\mathbf{u}\cdot\nabla p. \tag{17}$$

This equation governs the time evolution of the pressure. A way to memorize this equation is that, if $\gamma=1$, the equation will take the same form as a continuity equation, i.e.,

$$\frac{\partial p}{\partial t}=-\nabla\cdot(p\mathbf{u}). \tag{18}$$

2 Summary of resistive MHD equations

For the convenience of reference, the MHD equations discussed above are summarized here. The time evolution of the four quantities, namely $\mathbf{B}$, $\mathbf{u}$, p, and ρ, are governed respectively by the following four equations:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} / \mu_0, \tag{19}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + (\nabla \times \mathbf{B}) / \mu_0 \times \mathbf{B}, \tag{20}$$

$$\frac{\partial p}{\partial t} = -\gamma p \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p, \tag{21}$$

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla \rho. \tag{22}$$

Note that only Eq. (19) involves the resistivity η. When $\eta = 0$, the above system is called ideal MHD equations.

3 Linearized ideal MHD equation

Next, consider the linearized version of the ideal MHD equations. Use $\mathbf{u}_0$, $\mathbf{B}_0$, p_0, and ρ_0 to denote the equilibrium fluid velocity, magnetic field, plasma pressure, and mass density, respectively. Use $\mathbf{u}_1$, $\mathbf{B}_1$, p_1, and ρ_1 to denote the perturbed fluid velocity, magnetic field, plasma pressure, and mass density, respectively. Consider only the case that there is no equilibrium flow, i.e., $\mathbf{u}_0 = 0$. From Eq. (21), the linearized equation for the time evolution of the perturbed pressure is written as

$$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{u}_1 - \mathbf{u}_1 \cdot \nabla p_0. \tag{23}$$

The linearized momentum equation is

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla p_1 + \mu_0^{-1} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \mu_0^{-1} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1. \tag{24}$$

The linearized induction equation is

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0). \tag{25}$$

These three equations constitute a closed system for $\mathbf{u}_1$, $\mathbf{B}_1$, and p_1. Note that the linearized equation for the perturbed mass density ρ_1

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \nabla \cdot \mathbf{u}_1 - \mathbf{u}_1 \cdot \nabla \rho_0 \tag{26}$$

is not needed when solving the system of equations (23)-(25) because ρ_1 does not appear in equations (23)-(25).

3.1 Plasma displacement vector $\boldsymbol{\xi}$

In dealing with the linear case of MHD theory, it is convenient to introduce the plasma displacement vector $\boldsymbol{\xi}$, which is defined through the following equation

$$\mathbf{u}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t}. \tag{27}$$

Using the definition of $\boldsymbol{\xi}$ and the fact that the equilibrium quantities are independent of time, the linearized induction equation (25) is written

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{\partial}{\partial t}[\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)]. \tag{28}$$

Similarly, the equation for the perturbed pressure [Eq. (23)] is written

$$\frac{\partial p_1}{\partial t} = \frac{\partial}{\partial t}[-\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}]. \tag{29}$$

In terms of the displacement vector, the linearized momentum equation (24) is written

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla p_1 + \mu_0^{-1} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 + \mu_0^{-1} (\nabla \times \mathbf{B}_0) \times \mathbf{B}_1. \tag{30}$$

Equations (28), (29), and (30) constitute a closed system for $\mathbf{B}_1$, p_1, and $\boldsymbol{\xi}$.

3.2 Fourier transformation in time

A general perturbation can be written

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(\omega) e^{-i\omega t} d\omega, \tag{31}$$

where the coefficient $\hat{h}(\omega)$ is given by the Fourier transformation of $h(t)$, i.e.,

$$\hat{h}(\omega) = \int_{-\infty}^{\infty} h(t) e^{i\omega t} dt, \tag{32}$$

Using the definition of the Fourier transformtion, it is ready to prove that

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} h(t) e^{i\omega t} dt = -i\omega \hat{h}(\omega), \tag{33}$$

and

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} h(t) e^{i\omega t} dt = -\omega^2 \hat{h}(\omega). \tag{34}$$

Performing Fourier transformation (in time) on both sides of the the linearized momentum equation (30) and noting that the equilibrium quantities are all independent of time, we obtain

$$-\omega^2 \rho_0 \hat{\boldsymbol{\xi}} = -\nabla \hat{p}_1 + \mu_0^{-1} (\nabla \times \hat{\mathbf{B}}_1) \times \mathbf{B}_0 + \mu_0^{-1} (\nabla \times \mathbf{B}_0) \times \hat{\mathbf{B}}_1, \tag{35}$$

where use has been made of the property in Eq. (34). Similarly, the Fourier transformation of the equations of state (29) is written

$$\hat{p}_1 = -\hat{\boldsymbol{\xi}} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \hat{\boldsymbol{\xi}}, \tag{36}$$

and the Fourier transformation of Faraday's law (28) is written

$$\hat{\mathbf{B}}_1 = \nabla \times (\hat{\boldsymbol{\xi}} \times \mathbf{B}_0), \tag{37}$$

Equations (35)-(37) agree with Eqs. (12)-(14) in Cheng's paper[3]. They constitute a closed set of equations for $\hat{\boldsymbol{\xi}}$, $\hat{\mathbf{B}}_1$, and $\hat{p}_1$. In the next section, for notation ease, the hat on $\hat{\boldsymbol{\xi}}$, $\hat{\mathbf{B}}_1$, and $\hat{p}_1$ will be omitted, with the understanding that they are the Fourier transformations of the corresponding quantities.

3.3 Discrete frequency perturbation

In dealing with eigenmodes, we ususally encounter discrete frequency perturbations, i.e., $h(t)$ is periodic function of t so that they contian only discrete frequency components. In this case, the inverse Fourier transformtion in Eq. (31) is replaced by the Fourier series, i.e.,

$$h(t) = \sum_{j=-\infty}^{\infty} c(\omega_j) e^{-i\omega_j t}, \tag{38}$$

where the coefficients c_j are given by

$$c(\omega_j) = \frac{1}{T} \int_0^T h(t) e^{i\omega_j t} dt, \tag{39}$$

with $\omega_j = j2\pi/T$ and T being the period of $h(t)$ (T is larger enough so that $1/T$ is very small compared with frequency we are interested). The relation between $h(t)$ and $\hat{h}(\omega)$ given by Eqs. (33) and (34) also applies to the relation between $h(t)$ and $c(\omega)$, i.e.,

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} h(t) e^{i\omega t} dt = -i\omega c(\omega) \tag{40}$$

and

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} h(t) e^{i\omega t} dt = -\omega^2 c(\omega). \tag{41}$$

Using the transformtion given by Eq. (39) on Eqs. (28), (29), and (30), respectively, we obtain the same equation as Eqs. (35)-(37).

3.4 Eigenmodes

A general perturbation is given by Eq. (31), which is composed of infinit single frequency perturbation of the form $\hat{h}(\omega, \mathbf{r}) e^{-i\omega t}$. It is obvious that a general perturbation can also be considered to be composed of single frequency perturbation of the form

$$\hat{h}(\omega, \mathbf{r}) e^{-i\omega t} + \hat{h}(-\omega, \mathbf{r}) e^{i\omega t}, \tag{42}$$

(it is also obvious that the range of ω can be limited in $[0, +\infty]$ for this case). Because a physical quanty is always a real function, $h(t, \mathbf{r})$ in the above should be a real function. It is ready to prove that the Fourier transformation of $h(t, \mathbf{r})$ has the following symmetry in frequency domain:

$$\hat{h}(-\omega, \mathbf{r}) = \hat{h}^{\star}(\omega, \mathbf{r}). \tag{43}$$

Writing

$$\hat{h}(\omega, \mathbf{r}) = A e^{i\alpha}, \tag{44}$$

where A and α are real numbers, then the expression (42) is written

$$\begin{aligned}&\hat{h}(\omega, \mathbf{r})e^{-i\omega t} + \hat{h}(-\omega, \mathbf{r})e^{i\omega t}\\ =&\hat{h}(\omega, \mathbf{r})e^{-i\omega t} + \hat{h}^\star(\omega, \mathbf{r})e^{i\omega t}\\ =&\hat{h}(\omega, \mathbf{r})e^{-i\omega t} + [\hat{h}(\omega, \mathbf{r})e^{-i\omega t}]^\star\\ =&2A\cos[\alpha(\mathbf{r}) - \omega t]\end{aligned} \tag{45}$$

Using Eq. (45), the Fourier transformation is written

$$h(t, \mathbf{r}) = \int_0^\infty A(\omega, \mathbf{r})\cos[\alpha(\omega, \mathbf{r}) - \omega t]d\omega. \tag{46}$$

If $\hat{h}(\omega, \mathbf{r})$ satisfies the eigenmode equations (35)-(37), then it is ready to verify that $\hat{h}^\star(\omega, \mathbf{r})$ is also a solution to the equations. Since of Eq. (43), $\hat{h}(-\omega, \mathbf{r})$ is also a slution to the equations. Therefore $\hat{h}(\omega, \mathbf{r})e^{-i\omega t} + \hat{h}(-\omega, \mathbf{r})e^{i\omega t}$, i.e., $2A\cos[\alpha(\mathbf{r}) - \omega t]$, is a solution to the linear equations (28), (29), and (30). This tell us how to construct a real (physical) eigenmode from the complex funtion $\hat{h}(\omega, \mathbf{r})$, i.e., the real part of $\hat{h}(\omega, \mathbf{r})e^{-i\omega t}$ is a physical eigenmode. Note that it is the real part of $\hat{h}(\omega, \mathbf{r})e^{-i\omega t}$, instead of the real part of $\hat{h}(\omega, \mathbf{r})$, that is a physical eigenmode.

Note that ω is a real number by the definition of Fourier transformation. However, strictly speaking, the above Fourier transformation should be replaced by Laplace transformation. In this case, ω is a complex number. In the following, we will assume that we are using Laplace transformation, instead of Fourier transformation. In the following section, we will prove that the eigenvalue ω^2 of the ideal MHD system must be a real number.

3.5 Linear force operator

Using Eqs. (36) and (37) to eliminate p_1 and $\mathbf{B}_1$ from Eq. (35), we obtain

$$-\omega^2\rho_0\boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}), \tag{47}$$

where $\mathbf{F}(\boldsymbol{\xi})$, the linear force operator, is given by

$$\mathbf{F}(\boldsymbol{\xi}) = \nabla(\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\nabla\cdot\boldsymbol{\xi}) + (\nabla\times\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0))/\mu_0\times\mathbf{B}_0 + (\nabla\times\mathbf{B}_0)/\mu_0\times\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) \tag{48}$$

It can be proved that the linear force operator $\mathbf{F}(\boldsymbol{\xi})$ is self-adjoint (or Hermitian) (I have not proven this), i.e., for any two functions $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ that satisfy the same boundary condition, we have

$$\int\boldsymbol{\eta}_2\cdot\mathbf{F}(\boldsymbol{\eta}_1)d^3x = \int\boldsymbol{\eta}_1\cdot\mathbf{F}(\boldsymbol{\eta}_2)d^3x. \tag{49}$$

As a consequence of the self-adjointness, the eigenvalue, ω^2, must be a real number, which implies that ω itself is either purely real or purely imaginary. [Proof: Taking the complex conjugate of Eq. (47), we obtain

$$-\rho_0(\omega^2\boldsymbol{\xi})^\star = [\mathbf{F}(\boldsymbol{\xi})]^\star. \tag{50}$$

Note that the expression of $\mathbf{F}(\boldsymbol{\xi})$ given in Eq. (48) have the property $[\mathbf{F}(\boldsymbol{\xi})]^\star = \mathbf{F}(\boldsymbol{\xi}^\star)$, Using this, equation (50) is written

$$-\omega^{2\star}\rho_0\boldsymbol{\xi}^\star = \mathbf{F}(\boldsymbol{\xi}^\star), \tag{51}$$

Taking the scalar product of both sides of the above equation with $\boldsymbol{\xi}$ and integrating over the entire volume of the system, gives

$$-\omega^{2\star}\int\rho_0\boldsymbol{\xi}^\star\cdot\boldsymbol{\xi}d^3x = \int\mathbf{F}(\boldsymbol{\xi}^\star)\cdot\boldsymbol{\xi}d^3x, \tag{52}$$

Using the self-ajointness of $\mathbf{F}$, the above equation is written

$$-\omega^{2\star}\int\rho_0\boldsymbol{\xi}^\star\cdot\boldsymbol{\xi}d^3x = \int\mathbf{F}(\boldsymbol{\xi})\cdot\boldsymbol{\xi}^\star d^3x, \tag{53}$$

$$\Rightarrow -\omega^{2\star}\int\rho_0\boldsymbol{\xi}^\star\cdot\boldsymbol{\xi}d^3x = -\omega^2\int\rho_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}^\star d^3x, \tag{54}$$

$$\Rightarrow (\omega^2 - \omega^{2\star})\int\rho_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}^\star d^3x = 0 \tag{55}$$

Since $\int\rho_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}^\star d^3x$ is non-zero for any non-trivial eigenfunction, it follows from Eq. (55) that $\omega^2 = \omega^{2\star}$, i.e., ω^2 must be a real number, which implies that ω is either purely real or purely imaginary.] It can also be proved that two eigenfunctions with different ω^2 are orthogonal to each other. [Proof:

$$-\omega_m^{2\star}\rho_0\boldsymbol{\xi}_m^\star = \mathbf{F}(\boldsymbol{\xi}_m^\star), \tag{56}$$

$$-\omega_n^2\rho_0\boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n), \tag{57}$$

Taking the scalar product of both sides of Eq. (56) with $\boldsymbol{\xi}_n$ and integrating over the entire volume of the system, gives

$$\Rightarrow -\omega_m^{2\star}\int\rho_0\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n d^3x = \int \mathbf{F}(\boldsymbol{\xi}_m^\star)\cdot\boldsymbol{\xi}_n d^3x, \tag{58}$$

Taking the scalar product of both sides of Eq. (57) with $\boldsymbol{\xi}_m^\star$ and integrating over the entire volume of the system, gives

$$\Rightarrow -\omega_n^2\int\rho_0\boldsymbol{\xi}_n\cdot\boldsymbol{\xi}_m^\star d^3x = \int \mathbf{F}(\boldsymbol{\xi}_n)\cdot\boldsymbol{\xi}_m^\star d^3x, \tag{59}$$

Combining the above two equations, we obtain

$$(\omega_m^{2\star}-\omega_n^2)\int\rho_0\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n d^3x \;=\; \int [\mathbf{F}(\boldsymbol{\xi}_n)\cdot\boldsymbol{\xi}_m^\star - \mathbf{F}(\boldsymbol{\xi}_m^\star)\cdot\boldsymbol{\xi}_n]d^3x \tag{60}$$

Using the self-ajointness of $\mathbf{F}$, we know the right-hand side of Eq. (59) is zero. Thus Eq. (59) is written

$$(\omega_m^{2\star}-\omega_n^2)\int\rho_0\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n d^3x = 0. \tag{61}$$

Using $\omega_m^{2\star} = \omega_m^2$, the above equation is written

$$(\omega_m^2-\omega_n^2)\int\rho_0\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n d^3x = 0. \tag{62}$$

Since we assume $\omega_m^2 \neq \omega_n^2$, the above equation reduces to

$$\int\rho_0\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n d^3x = 0, \tag{63}$$

i.e., $\boldsymbol{\xi}_m^\star$ and $\boldsymbol{\xi}_n$ are orthogonal to each other.]

4 Components of MHD equations in toroidal geometry

Next, we consider the form of the linearized MHD equations in toroidal devices (e.g. tokamak). In these devices, there exist magnetic surfaces. The motion of plasma along the surface and perpendicular to the surface are very different. Thus, it is useful to decompose the perturbed quantities into components lying on the surface and perpendicular to the surface. Following Ref. [3, 4], we write the displacement vector and perturbed magnetic field as

$$\boldsymbol{\xi} = \xi_\psi\frac{\nabla\Psi}{|\nabla\Psi|^2} + \xi_s\frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2} + \xi_b\frac{\mathbf{B}_0}{B_0^2}, \tag{64}$$

and

$$\mathbf{B}_1 = Q_\psi\frac{\nabla\Psi}{|\nabla\Psi|^2} + Q_s\frac{(\mathbf{B}_0\times\nabla\Psi)}{|\nabla\Psi|^2} + Q_b\frac{\mathbf{B}_0}{B^2}, \tag{65}$$

where $\Psi = \Psi_{\mathrm{pol}}/(2\pi) + C$ with Ψ_{pol} the poloidal magnetic flux within a magnetic surface and C being an arbitrary constant and (In the process of deriving the eigenmode equation, we do not need the specific definition of Ψ. What we need is only that $\nabla\Psi$ is a vector in the direction of ∇p_0 and thus $\nabla\Psi$ is perpendicular to both $\mathbf{B}_0$ and $\mathbf{J}_0$). Taking scalar product of the above two equations with $\nabla\Psi$, $\mathbf{B}_0\times\nabla\Psi$, and $\mathbf{B}_0$, respectively, we obtain

$$\xi_\psi = \boldsymbol{\xi}\cdot\nabla\Psi,\;\; \xi_s = \boldsymbol{\xi}\cdot\left(\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2}\right),\;\; \xi_b = \boldsymbol{\xi}\cdot\mathbf{B}_0, \tag{66}$$

$$Q_\psi = \mathbf{B}_1\cdot\nabla\Psi,\;\; Q_s = \mathbf{B}_1\cdot\left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right),\;\; Q_b = \mathbf{B}_1\cdot\mathbf{B}_0. \tag{67}$$

Next, we derive the component equations for the induction equation (37) and momentum equation (35). The derivation is straightforward but tedious. Those who are not interested in these details can skip them and read directly Sec. 4.7 for the final form of the component equations.

4.1 $\nabla\Psi$ component of induction equation

The induction equation is given by Eq. (37), i.e.,

$$\mathbf{B}_1 = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0). \tag{68}$$

Next, consider the $\nabla\Psi$ component of the above equation. Taking scalar product of the above equation with $\nabla\Psi$, we obtain

$$\mathbf{B}_1\cdot\nabla\Psi=\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\cdot\nabla\Psi. \tag{69}$$

$$\Rightarrow Q_\psi=\nabla\times\left[\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\times\mathbf{B}_0+\frac{\xi_s}{B^2}(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0\right]\cdot\nabla\Psi \tag{70}$$

$$\Rightarrow Q_\psi=\nabla\times\left[\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\times\mathbf{B}_0+\xi_s\nabla\Psi\right]\cdot\nabla\Psi \tag{71}$$

$$\Rightarrow Q_\psi=\left[\nabla\times\left(\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\times\mathbf{B}_0\right)+\nabla\times(\xi_s\nabla\Psi)\right]\cdot\nabla\Psi \tag{72}$$

$$\Rightarrow Q_\psi=\left[\nabla\times\left(\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\times\mathbf{B}_0\right)+\nabla\xi_s\times\nabla\Psi\right]\cdot\nabla\Psi \tag{73}$$

$$\Rightarrow Q_\psi=\left[\nabla\times\left(\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\times\mathbf{B}_0\right)\right]\cdot\nabla\Psi \tag{74}$$

$$\Rightarrow Q_\psi=\left[(\mathbf{B}_0\cdot\nabla)\left(\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\right)-\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\cdot\nabla\mathbf{B}_0\right]\cdot\nabla\Psi \tag{75}$$

$$Q_\psi=\left[\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)+\nabla\Psi(\mathbf{B}_0\cdot\nabla)\left(\frac{\xi_\psi}{|\nabla\Psi|^2}\right)-\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\cdot\nabla\mathbf{B}_0\right]\cdot\nabla\Psi \tag{76}$$

$$Q_\psi=\left[\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)+\left(\frac{1}{|\nabla\Psi|^2}\right)\nabla\Psi(\mathbf{B}_0\cdot\nabla)\xi_\psi+\xi_\psi\nabla\Psi(\mathbf{B}_0\cdot\nabla)\frac{1}{|\nabla\Psi|^2}-\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\cdot\nabla\mathbf{B}_0\right]\cdot\nabla\Psi \tag{77}$$

$$Q_\psi=\left[\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)\cdot\nabla\Psi+(\mathbf{B}_0\cdot\nabla)\xi_\psi+\xi_\psi|\nabla\Psi|^2(\mathbf{B}_0\cdot\nabla)\frac{1}{|\nabla\Psi|^2}-\frac{\xi_\psi}{|\nabla\Psi|^2}(\nabla\Psi\cdot\nabla\mathbf{B}_0)\cdot\nabla\Psi\right] \tag{78}$$

Excluding $(\mathbf{B}_0\cdot\nabla)\xi_\psi$ terms, the terms on the right hand side (r.h.s) of the above equation can be written

$$\begin{aligned}
&\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)\cdot\nabla\Psi+\xi_\psi|\nabla\Psi|^2(\mathbf{B}_0\cdot\nabla)\frac{1}{|\nabla\Psi|^2}-\frac{\xi_\psi}{|\nabla\Psi|^2}(\nabla\Psi\cdot\nabla\mathbf{B}_0)\cdot\nabla\Psi\\
&=\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)\cdot\nabla\Psi-\frac{1}{|\nabla\Psi|^2}\xi_\psi(\mathbf{B}_0\cdot\nabla)|\nabla\Psi|^2-\frac{\xi_\psi}{|\nabla\Psi|^2}(\nabla\Psi\cdot\nabla\mathbf{B}_0)\cdot\nabla\Psi\\
&=\frac{\xi_\psi}{|\nabla\Psi|^2}(\mathbf{B}_0\cdot\nabla)(\nabla\Psi)\cdot\nabla\Psi-\frac{\xi_\psi}{|\nabla\Psi|^2}2\nabla\Psi\cdot(\mathbf{B}_0\cdot\nabla)\nabla\Psi-\frac{\xi_\psi}{|\nabla\Psi|^2}(\nabla\Psi\cdot\nabla\mathbf{B}_0)\cdot\nabla\Psi\\
&=-\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\cdot(\mathbf{B}_0\cdot\nabla)\nabla\Psi-\frac{\xi_\psi}{|\nabla\Psi|^2}(\nabla\Psi\cdot\nabla\mathbf{B}_0)\cdot\nabla\Psi\\
&=-\frac{\xi_\psi}{|\nabla\Psi|^2}\nabla\Psi\cdot[(\mathbf{B}_0\cdot\nabla)\nabla\Psi+\nabla\Psi\cdot\nabla\mathbf{B}_0]
\end{aligned} \tag{79}$$

Using $\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{B}\cdot\nabla\mathbf{A}+\mathbf{A}\cdot\nabla\mathbf{B}+\mathbf{A}\times\nabla\times\mathbf{B}+\mathbf{B}\times\nabla\times\mathbf{A}$, we obtain

$$\nabla(\nabla\Psi\cdot\mathbf{B}_0) = \mathbf{B}_0\cdot\nabla\nabla\Psi+\nabla\Psi\cdot\nabla\mathbf{B}_0+\nabla\Psi\times\nabla\times\mathbf{B}_0 \tag{80}$$

$$0 = \nabla\Psi\cdot\nabla(\nabla\Psi\cdot\mathbf{B}_0)=\nabla\Psi\cdot[\mathbf{B}_0\cdot\nabla\nabla\Psi+\nabla\Psi\cdot\nabla\mathbf{B}_0] \tag{81}$$

The r.h.s of the above equation is exactly the term appearing on the right-hand side of Eq. (79). Thus we obtain

$$Q_\psi=\mathbf{B}_0\cdot\nabla\xi_\psi, \tag{82}$$

which agrees with Eq. (20) in Cheng's paper[3].

4.2 $B_0\times\nabla\Psi$ component of induction equation

The $\mathbf{B}_0\times\nabla\Psi$ component of the induction equation is given by

$$\mathbf{B}_1\cdot\mathbf{B}_0\times\nabla\Psi=\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)\cdot(\mathbf{B}_0\times\nabla\Psi), \tag{83}$$

which can be further written as

$$\begin{aligned}
B_0^2Q_s &= \nabla\times\left(\xi_\psi\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}+\xi_s\nabla\Psi\right)\cdot(\mathbf{B}_0\times\nabla\Psi)\\
&= \left[\nabla\times\left(\xi_\psi\,\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}\right)+\nabla\xi_s\times\nabla\Psi\right]\cdot(\mathbf{B}_0\times\nabla\Psi)\\
&= \left[\xi_\psi\nabla\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}+\nabla\xi_\psi\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}+\nabla\xi_s\times\nabla\Psi\right]\cdot(\mathbf{B}_0\times\nabla\Psi)\\
&= \left[\xi_\psi\nabla\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}+\nabla\xi_s\times\nabla\Psi\right]\cdot(\mathbf{B}_0\times\nabla\Psi)\\
&= -|\nabla\psi|^2S\xi_\psi+(\nabla\xi_s\times\nabla\Psi)\cdot(\mathbf{B}_0\times\nabla\Psi),
\end{aligned} \tag{84}$$

where

$$S=\left(\nabla\times\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2}\right)\cdot\frac{(\mathbf{B}_0\times\nabla\Psi)}{|\nabla\Psi|^2}, \tag{85}$$

is the negative local magnetic shear. Using $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})=(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D})-(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$, equation (84) is written as

$$B_0^2 Q_s = -|\nabla\Psi|^2 S\xi_\psi + |\nabla\Psi|^2 \mathbf{B}_0\cdot\nabla\xi_s, \tag{86}$$

$$\Rightarrow Q_s = \left(\frac{|\nabla\Psi|}{B_0}\right)^2 (\mathbf{B}_0\cdot\nabla\xi_s - S\xi_\psi). \tag{87}$$

Eq. (87) agrees with Eq. (21) in Cheng's paper[3].

4.3 B_0 component of induction equation

The component of the induction equation in the direction of $\mathbf{B}_0$ is written as

$$\mathbf{B}_0\cdot\mathbf{B}_1 = \mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) \tag{88}$$

$$\Rightarrow Q_b = \mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) \tag{89}$$

The term on right-hand side of the above equation is written as

$$\begin{aligned}\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) &= \nabla\times\left(\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}\xi_\psi + \frac{\xi_s}{B^2}(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0\right)\\ &= \nabla\times\left(\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}\xi_\psi + \xi_s\nabla\Psi\right)\\ &= \xi_\psi\nabla\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2} + \nabla\xi_\psi\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2} + \nabla\xi_s\times\nabla\Psi. \end{aligned}\tag{90}$$

Using this, the right-hand side of Eq. (89) is written as

$$\begin{aligned}\mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) &= \xi_\psi\mathbf{B}_0\cdot\nabla\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2} + \left(\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2}\times\mathbf{B}_0\right)\cdot\nabla\xi_\psi + (\nabla\Psi\times\mathbf{B}_0)\cdot\nabla\xi_s\\ &= \xi_\psi\mathbf{B}_0\cdot\nabla\times\frac{\nabla\Psi\times\mathbf{B}_0}{|\nabla\Psi|^2} + \left(-\frac{B_0^2}{|\nabla\Psi|^2}\nabla\Psi\right)\cdot\nabla\xi_\psi + (\nabla\Psi\times\mathbf{B}_0)\cdot\nabla\xi_s \qquad (91)\\ &= \xi_\psi\mathbf{B}_0\cdot\left(-\mathbf{B}_0\nabla\cdot\frac{\nabla\Psi}{|\nabla\Psi|^2} + \mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2} - \frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right) + \left(-\frac{B_0^2}{|\nabla\Psi|^2}\nabla\Psi\right)\cdot\nabla\xi_\psi + (\nabla\Psi\times\mathbf{B}_0)\cdot\\ &\quad\nabla\xi_s\\ &= \xi_\psi\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2} - \frac{\nabla\Psi\cdot\nabla\mathbf{B}_0}{|\nabla\Psi|^2}\right) - \xi_\psi B_0^2\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) - \frac{B_0^2}{|\nabla\Psi|^2}\nabla\Psi\cdot\nabla\xi_\psi - (\mathbf{B}_0\times\nabla\Psi)\cdot\\ &\quad\nabla\xi_s \end{aligned}\tag{92}$$

Before we try to simplify the above equation, we derive the expression for the divergence of $\boldsymbol{\xi}$, which is written as

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi} &= \nabla\cdot\left[\frac{\nabla\Psi}{|\nabla\Psi|^2}\xi_\psi + \frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2}\xi_s + \frac{\mathbf{B}_0}{B_0^2}\xi_b\right]\\ &= \frac{\nabla\Psi\cdot\nabla\xi_\psi}{|\nabla\Psi|^2} + \xi_\psi\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) + \frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2}\cdot\nabla\xi_s + \xi_s\nabla\cdot\frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2} + \mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right)\end{aligned}\tag{93}$$

It can be proved that the fourth term of the above equation can be written as (refer to (10.8) for the proof)

$$\nabla\cdot\frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2} = -2\frac{1}{B_0^2}\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi). \tag{94}$$

Then Eq. (93) is written as

$$\nabla\cdot\boldsymbol{\xi} = \frac{\nabla\Psi\cdot\nabla\xi_\psi}{|\nabla\Psi|^2} + \xi_\psi\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) + \frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2}\cdot\nabla\xi_s - \xi_s\frac{2}{B_0^2}\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi) + \mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right) \tag{95}$$

Eq. (95) agrees with Eq. (23) in Cheng's paper, but a B_0^2 factor is missed in the fourth term of Cheng's equation[3]. Using Eq. (95), Eq. (92) is written as

$$\mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) = \xi_\psi\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2} - \frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right) - B_0^2\nabla\cdot\boldsymbol{\xi} - 2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\Psi)\xi_s + B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right) \tag{96}$$

It can be proved that

$$\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2} - \frac{\nabla\Psi\cdot\nabla\mathbf{B}_0}{|\nabla\Psi|^2}\right) = -2\boldsymbol{\kappa}\cdot\nabla\Psi\frac{B^2}{|\nabla\Psi|^2} + \mu_0\frac{dp_0}{d\Psi}. \tag{97}$$

(Refer to Sec. 10.10 for the proof.) Then Eq. (96) is written as

$$\mathbf{B}_0\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0) = \xi_\psi\left(-\frac{2B_0^2}{|\nabla\Psi|^2}\nabla\Psi\cdot\boldsymbol{\kappa} + \mu_0\frac{dp_0}{d\Psi}\right) - B_0^2\nabla\cdot\boldsymbol{\xi} - 2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\Psi)\xi_s + B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right), \tag{98}$$

and the component of the induction equation in the direction of $\mathbf{B}_0$ [Eq. (89)] is finally written as

$$Q_b = \xi_\psi\left(-\frac{2B_0^2}{|\nabla\Psi|^2}\nabla\Psi\cdot\boldsymbol{\kappa} + \mu_0\frac{dp_0}{d\Psi}\right) - B_0^2\nabla\cdot\boldsymbol{\xi} - 2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\Psi)\xi_s + B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right) \tag{99}$$

Eq. (99) agrees with Eq. (22) in Cheng's paper[3].

4.4 Component of the momentum equation parallel to B_0

The three components of the linearized momentum equation can be obtained by taking scalar product of Eq. (35) with $\nabla\Psi$, $\mathbf{B}_0\times\nabla\psi$, and $\mathbf{B}_0$, respectively. We first consider the $\mathbf{B}_0$ component of the momentum equation, which is written as

$$-\omega^2\rho_0\boldsymbol{\xi}\cdot\mathbf{B}_0 = -\mathbf{B}_0\cdot\nabla p_1 + \mathbf{B}_0\cdot[\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1], \tag{100}$$

i.e.,

$$\omega^2\rho_0\xi_b = \mathbf{B}_0\cdot\nabla p_1 + \mathbf{J}_0\cdot(\mathbf{B}_0\times\mathbf{B}_1). \tag{101}$$

The last term on the right-hand side of Eq. (101) can be written as

$$\begin{aligned}
\mathbf{J}_0\cdot(\mathbf{B}_0\times\mathbf{B}_1) &= \mathbf{J}_0\cdot\mathbf{B}_0\times\left[\frac{Q_\psi}{|\nabla\Psi|^2}\nabla\Psi + \frac{Q_s}{|\nabla\Psi|^2}(\mathbf{B}_0\times\nabla\Psi)\right]\\
&= \mathbf{J}_0\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}\mathbf{B}_0\times\nabla\Psi + \frac{Q_s}{|\nabla\Psi|^2}\mathbf{B}_0\times(\mathbf{B}_0\times\nabla\Psi)\right]\\
&= \mathbf{J}_0\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}\mathbf{B}_0\times\nabla\Psi - \frac{Q_s}{|\nabla\Psi|^2}B_0^2\nabla\Psi\right]\\
&= \mathbf{J}_0\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}\mathbf{B}_0\times\nabla\Psi\right] - 0\\
&= \nabla\Psi\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}\mathbf{J}_0\times\mathbf{B}_0\right]\\
&= \nabla\Psi\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}\nabla p_0\right]\\
&= \nabla\Psi\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}p_0'\nabla\Psi\right]\\
&= Q_\psi p_0'.
\end{aligned} \tag{102}$$

where $p_0'\equiv dp_0/d\Psi$. Using Eq. (82), i.e., $Q_\psi = \mathbf{B}_0\cdot\nabla\xi_\psi$, the above equation is written as

$$\mathbf{J}_0\cdot\mathbf{B}_0\times\mathbf{B}_1 = (\mathbf{B}_0\cdot\nabla\xi_\psi)p_0' \tag{103}$$

Substituting Eq. (103) into Eq. (101) gives

$$\omega^2\rho_0\xi_b = \mathbf{B}_0\cdot\nabla p_1 + (\mathbf{B}_0\cdot\nabla\xi_\psi)p_0' \tag{104}$$

Using $\mathbf{B}_0\cdot\nabla p_0' = 0$ in the above equation gives

$$\omega^2\rho_0\xi_b = \mathbf{B}_0\cdot\nabla(p_1 + p_0'\xi_\psi), \tag{105}$$

which agrees with Eq. (19) in Cheng's paper[3].

4.5 $\nabla\Psi$ component of momentum equation

Next we consider the radial component of the momentum equation. Taking scalar product of the momentum equation with $\nabla\Psi$, we obtain

$$-\omega^2\rho_0\xi_\psi = -\nabla\Psi\cdot\nabla p_1 + \nabla\Psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0 + \nabla\Psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1. \tag{106}$$

After some algebra (the details are given in Sec. (10.5)), Eq. (106) is written

$$\begin{aligned}
-\omega^2\rho_0\xi_\psi &= -\nabla\Psi\cdot\nabla P_1 + \mu_0^{-1}|\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{Q_\psi}{|\nabla\Psi|^2}\right)\\
&\quad + (\mu_0^{-1}|\nabla\Psi|^2 S - \mathbf{B}_0\cdot\mathbf{J}_0)Q_s + 2\mu_0^{-1}\boldsymbol{\kappa}\cdot\nabla\Psi Q_b,
\end{aligned} \tag{107}$$

where

$$P_1 \equiv p_1 + \frac{\mathbf{B}_1\cdot\mathbf{B}_0}{\mu_0}, \tag{108}$$

and $\boldsymbol{\kappa}\equiv\mathbf{b}\cdot\nabla\mathbf{b}$ is the magnetic field curvature with $\mathbf{b}=\mathbf{B}_0/B_0$ the unit vector along equilibrium magnetic field. Equation (107) agrees with Eq. (17) in Cheng's paper[3]. In passing, let us examine the physical meaning of P_1 defined by (108). In linear approximation, we have

$$\begin{aligned}
B^2 &= |\mathbf{B}_0 + \mathbf{B}_1|^2\\
&= B_0^2 + B_1^2 + 2\mathbf{B}_0\cdot\mathbf{B}_1\\
&\approx B_0^2 + 2\mathbf{B}_0\cdot\mathbf{B}_1
\end{aligned} \tag{109}$$

This indicates the perturbation in the square of the magnetic strength is $2\mathbf{B}_0\cdot\mathbf{B}_1$. Therefore, the perturbation in magnetic pressure is written

$$\Delta P_{\text{mag}} \equiv \frac{\Delta(B^2)}{2\mu_0} = \frac{2\mathbf{B}_0\cdot\mathbf{B}_1}{2\mu_0} = \frac{\mathbf{B}_0\cdot\mathbf{B}_1}{\mu_0}, \tag{110}$$

which indicate P_1 defined by Eq. (108) is the total perturbation in the thermal and magnetic pressure.

4.6 $B_0 \times \nabla\Psi$ component of momentum equation

The $\mathbf{B}_0 \times \nabla\Psi$ component of the linearized momentum equation is written

$$-\omega^2\rho_0|\nabla\Psi|^2\xi_s = -(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla p_1 + (\mathbf{B}_0\times\nabla\Psi)\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0 + (\mathbf{B}_0\times\nabla\Psi)\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1. \quad (111)$$

The last term of Eq. (111) is written

$$\begin{aligned}(\mathbf{B}_0\times\nabla\Psi)\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1 &= (\mathbf{B}_0\times\nabla\Psi)\cdot\mathbf{J}_0\times\mathbf{B}_1\\ &= -\mathbf{J}_0\cdot(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_1\\ &= -\mathbf{J}_0\cdot(\mathbf{B}_0\times\nabla\Psi)\times\left[\frac{Q_\psi}{|\nabla\Psi|^2}\nabla\Psi+\frac{Q_b}{B_0^2}\mathbf{B}_0\right]\\ &= -\mathbf{J}_0\cdot\left[\frac{Q_\psi}{|\nabla\Psi|^2}(0-|\nabla\Psi|^2\mathbf{B}_0)+\frac{Q_b}{B_0^2}(B_0^2\nabla\Psi-0)\right]\\ &= -\mathbf{J}_0\cdot[-Q_\psi\mathbf{B}_0+Q_b\nabla\Psi]\\ &= \mathbf{B}_0\cdot\mathbf{J}_0Q_\psi. \end{aligned} \quad (112)$$

Using Eq. (335), i.e.,

$$\nabla\times\mathbf{B}_1 = \nabla\frac{Q_\psi}{|\nabla\Psi|^2}\times\nabla\Psi - \left(\mathbf{B}_0\cdot\nabla\frac{Q_s}{|\nabla\Psi|^2}\right)\nabla\Psi + \left(\nabla\frac{Q_s}{|\nabla\Psi|^2}\cdot\nabla\Psi\right)\mathbf{B}_0 + \frac{Q_s}{|\nabla\Psi|^2}\nabla\times(\mathbf{B}_0\times\nabla\Psi) + \nabla\frac{Q_b}{B_0^2}\times\mathbf{B}_0 + \frac{Q_b}{B_0^2}\mu_0\mathbf{J}_0. \quad (113)$$

the second last term on the right-hand side of Eq. (111) is written

$$\begin{aligned}(\mathbf{B}_0\times\nabla\Psi)\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0 &= \mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\left[\nabla\frac{Q_\psi}{|\nabla\Psi|^2}\times\nabla\Psi-\left(\mathbf{B}_0\cdot\nabla\frac{Q_s}{|\nabla\Psi|^2}\right)\nabla\Psi+\frac{Q_s}{|\nabla\Psi|^2}\nabla\times(\mathbf{B}_0\times\nabla\Psi)+\right.\\ &\quad\left.\nabla\frac{Q_b}{B_0^2}\times\mathbf{B}_0+\frac{Q_b}{B_0^2}\mu_0\mathbf{J}_0\right]\times\mathbf{B}_0\\ &= \mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\left[\left(\mathbf{B}_0\cdot\nabla\frac{Q_\psi}{|\nabla\Psi|^2}\right)\nabla\Psi-\left(\mathbf{B}_0\cdot\nabla\frac{Q_s}{|\nabla\Psi|^2}\right)\nabla\Psi\times\mathbf{B}_0+\frac{Q_s}{|\nabla\Psi|^2}\nabla\times\right.\\ &\quad\left.(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0+\nabla\frac{Q_b}{B_0^2}\times\mathbf{B}_0\times\mathbf{B}_0\right]\\ &= \mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\left[-\left(\mathbf{B}_0\cdot\nabla\frac{Q_s}{|\nabla\Psi|^2}\right)\nabla\Psi\times\mathbf{B}_0+\frac{Q_s}{|\nabla\Psi|^2}\nabla\times(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0+\right.\\ &\quad\left.\mathbf{B}_0(\mathbf{B}_0\cdot\nabla\frac{Q_b}{B_0^2})-B_0^2\nabla\frac{Q_b}{B_0^2}\right]\\ &= \mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\left[-\left(\mathbf{B}_0\cdot\nabla\frac{Q_s}{|\nabla\Psi|^2}\right)\nabla\Psi\times\mathbf{B}_0+\frac{Q_s}{|\nabla\Psi|^2}\nabla\times(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0-\right.\\ &\quad\left.B_0^2\nabla\frac{Q_b}{B_0^2}\right] \quad (114)\\ &= \mu_0^{-1}\underbrace{B_0^2(\mathbf{B}_0\cdot\nabla Q_s)}+\mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\left[Q_s\left(\mathbf{B}_0\cdot\nabla\frac{1}{|\nabla\Psi|^2}\right)\mathbf{B}_0\times\nabla\Psi+\frac{Q_s}{|\nabla\Psi|^2}\nabla\times\right.\\ &\quad\left.(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0-B_0^2\nabla\frac{Q_b}{B_0^2}\right] \quad (115)\end{aligned}$$

The term $\nabla\times(\mathbf{B}_0\times\nabla\Psi)$ in the above equation is written as

$$\begin{aligned}\nabla\times(\mathbf{B}_0\times\nabla\Psi) &= -\nabla\times(\nabla\Psi\times\mathbf{B}_0)\\ &= -[-\mathbf{B}_0(\nabla\cdot\nabla\Psi)+\mathbf{B}_0\cdot\nabla\nabla\Psi-\nabla\Psi\cdot\nabla\mathbf{B}_0]\\ &= \mathbf{B}_0(\nabla^2\Psi)-(\mathbf{B}_0\cdot\nabla)\nabla\Psi+\nabla\Psi\cdot\nabla\mathbf{B}_0\\ \Rightarrow\frac{Q_s}{|\nabla\psi|^2}\nabla\times(\mathbf{B}_0\times\nabla\Psi)\times\mathbf{B}_0 &= \frac{Q_s}{|\nabla\Psi|^2}[-(\mathbf{B}_0\cdot\nabla)\nabla\Psi\times\mathbf{B}_0+\nabla\Psi\cdot\nabla\mathbf{B}_0\times\mathbf{B}_0] \end{aligned} \quad (116)$$

Gathering terms involving Q_s, excluding the first term, in expression (115) gives

$$(\mathbf{B}_0\times\nabla\Psi)\cdot\left[Q_s\left(\mathbf{B}_0\cdot\nabla\frac{1}{|\nabla\Psi|^2}\right)\mathbf{B}_0\times\nabla\Psi+\frac{Q_s}{|\nabla\Psi|^2}[-(\mathbf{B}_0\cdot\nabla\nabla\Psi)\times\mathbf{B}_0+(\nabla\Psi\cdot\nabla\mathbf{B}_0)\times\mathbf{B}_0]\right],$$

which can be proved to be zero (refer to Sec. 10.6 for the proof). Gathering terms involving Q_b in expression (115) gives

$$\begin{aligned}&(\mathbf{B}_0\times\nabla\Psi)\cdot\left[-B_0^2\nabla\frac{Q_b}{B_0^2}\right]\\ =&(\mathbf{B}_0\times\nabla\Psi)\cdot\left[-B_0^2Q_b\nabla\frac{1}{B_0^2}-\nabla Q_b\right]\\ =&\underbrace{-(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla Q_b}-B_0^2Q_b(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla\frac{1}{B_0^2}\end{aligned}$$

It can be proved that the second term of the above expression is equal to $2\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi)Q_b$ (refer to Sec. 10.9 for details). Thus, from Eq. (114), we obtain

$$(\mathbf{B}_0\times\nabla\Psi)\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0 = \mu_0^{-1}B_0^2(\mathbf{B}_0\cdot\nabla Q_s)-\mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla Q_b+2\mu_0^{-1}\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi)Q_b \quad (117)$$

Using Eqs. (112) and (117) in Eq. (111) yields

$$-\omega^2\rho_0|\nabla\Psi|^2\xi_s = -(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla p_1 + \mu_0^{-1}B_0^2(\mathbf{B}_0\cdot\nabla Q_s) - \mu_0^{-1}(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla Q_b + 2\mu_0^{-1}\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi)Q_b + \mathbf{B}_0\cdot\mathbf{J}_0 Q_\psi. \tag{118}$$

Using $Q_b=\mathbf{B}_1\cdot\mathbf{B}_0$, the above equation can be arranged as

$$-\omega^2\rho_0|\nabla\Psi|^2\xi_s = -(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla(p_1+\mu_0^{-1}\mathbf{B}_1\cdot\mathbf{B}_0) + \mu_0^{-1}B_0^2\mathbf{B}_0\cdot\nabla Q_s + 2\mu_0^{-1}\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi)Q_b + \mathbf{B}_0\cdot\mathbf{J}_0 Q_\psi, \tag{119}$$

which agrees with Eq. (18) in Cheng's paper[3].

4.7 Summary of component equations

For the ease of reference, Eqs. (29), (82), (87), (99), (105), (76), and (119) are repeated here:

$$p_1 + p_0'\xi_\psi = -\gamma p_0\nabla\cdot\boldsymbol{\xi}, \tag{120}$$

$$Q_\psi = \mathbf{B}_0\cdot\nabla\xi_\psi, \tag{121}$$

$$Q_s = \frac{|\nabla\Psi|^2}{B_0^2}(\mathbf{B}_0\cdot\nabla\xi_s - S\xi_\psi), \tag{122}$$

$$Q_b = \left(-\frac{2B_0^2}{|\nabla\Psi|^2}\kappa_\psi + \mu_0 p_0'\right)\xi_\psi - B_0^2\nabla\cdot\boldsymbol{\xi} - 2\kappa_s B_0^2\xi_s + B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\xi_b}{B_0^2}\right) \tag{123}$$

$$\begin{aligned}-\omega^2\rho_0\xi_\psi &= -\nabla\Psi\cdot\nabla P_1 + \mu_0^{-1}|\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{Q_\psi}{|\nabla\Psi|^2}\right)\\ &\quad +(\mu_0^{-1}|\nabla\Psi|^2 S - B_0^2\sigma)Q_s + 2\mu_0^{-1}\kappa_\psi Q_b.\end{aligned} \tag{124}$$

$$-\omega^2\rho_0|\nabla\Psi|^2\xi_s = -(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla P_1 + B_0^2\sigma Q_\psi + \mu_0^{-1}B_0^2\mathbf{B}_0\cdot\nabla Q_s + 2\mu_0^{-1}\kappa_s B_0^2 Q_b, \tag{125}$$

$$\omega^2\rho_0\xi_b = \mathbf{B}_0\cdot\nabla(p_1 + p_0'\xi_\psi), \tag{126}$$

where $p_0'\equiv dp_0/d\Psi$, $\sigma\equiv\mathbf{B}_0\cdot\mathbf{J}_0/B_0^2$, $\kappa_s\equiv\boldsymbol{\kappa}\cdot(\mathbf{B}_0\times\nabla\Psi)/B_0^2$, which is usually called the geodesic curvature, $\kappa_\psi\equiv\boldsymbol{\kappa}\cdot\nabla\Psi$, which is usually called the normal curvature.

4.8 Eigenmode equations using $(P_1, \xi_\psi, \xi_s, \nabla\cdot\boldsymbol{\xi})$ as variables

Using Eqs. (121) and (122) to eliminate Q_ψ, Q_s, Eqs. (124) and (125) are written, respectively, as

$$\begin{aligned}\nabla\Psi\cdot\nabla P_1 &= \omega^2\rho_0\xi_\psi + \mu_0^{-1}|\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla\xi_\psi}{|\nabla\Psi|^2}\right)\\ &+ (\mu_0^{-1}|\nabla\Psi|^2 S - B_0^2\sigma)\frac{|\nabla\Psi|^2}{B_0^2}(\mathbf{B}_0\cdot\nabla\xi_s - S\xi_\psi)\\ &+ 2\mu_0^{-1}\kappa_\psi Q_b.\end{aligned} \tag{127}$$

and

$$\begin{aligned}-\omega^2\rho_0|\nabla\Psi|^2\xi_s &= -(\mathbf{B}_0\times\nabla\Psi)\cdot\nabla P_1 + B_0^2\sigma\mathbf{B}_0\cdot\nabla\xi_\psi + \mu_0^{-1}B_0^2\mathbf{B}_0\cdot\nabla\left[\frac{|\nabla\Psi|^2}{B_0^2}(\mathbf{B}_0\cdot\nabla\xi_s - S\xi_\psi)\right]\\ &+ 2\mu_0^{-1}B_0^2\kappa_s Q_b\end{aligned} \tag{128}$$

Following Ref. [3], we will express the final eigenmodes equations in terms of the following four variables: P_1, ξ_ψ, ξ_s, and $\nabla\cdot\boldsymbol{\xi}$. In order to achieve this, we need to eliminate unwanted variables. Since we will use $P_1\equiv p_1+\mathbf{B}_0\cdot\mathbf{B}_1/\mu_0$ instead of p_1 as one of the variables. we write the equation (120) for the perturbed pressure in terms of P_1 variable, which gives

$$P_1 - \mu_0^{-1}Q_b + p_0'\xi_\psi = -\gamma p_0\nabla\cdot\boldsymbol{\xi}. \tag{129}$$

Using (129) to eliminate Q_b in Eq. (128), we obtain

$$\begin{aligned}-\frac{\omega^2\rho_0|\nabla\Psi|^2}{B_0^2}\xi_s &= -\left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right)\cdot\nabla P_1 + \sigma\mathbf{B}_0\cdot\nabla\xi_\psi + \mu_0^{-1}\mathbf{B}_0\cdot\nabla\left[\frac{|\nabla\Psi|^2}{B_0^2}(\mathbf{B}_0\cdot\nabla\xi_s - S\xi_\psi)\right]\\ &+ 2\kappa_s\gamma p_0\nabla\cdot\boldsymbol{\xi} + 2\kappa_s P_1 + 2\kappa_s p_0'\xi_\psi\end{aligned} \tag{130}$$

Using pressure equation (120) to eliminate p_1 in Eq. (126), we obtain

$$\omega^2\rho_0\xi_b = -\gamma p_0\mathbf{B}_0\cdot\nabla(\nabla\cdot\boldsymbol{\xi}), \tag{131}$$

which can be used in Eq. (123) to eliminate ξ_b, yielding

$$Q_b = \xi_\psi\left(-\frac{2B_0^2}{|\nabla\Psi|^2}\kappa_\psi + \mu_0 p_0'\right) - B_0^2\nabla\cdot\boldsymbol{\xi} - 2\kappa_s B_0^2\xi_s - \frac{\gamma p_0}{\omega^2\rho_0}B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla(\nabla\cdot\boldsymbol{\xi})}{B_0^2}\right). \tag{132}$$

Using Eq. (132) to eliminate Q_b in Eq. (129), we obtain

$$P_1+\mu_0^{-1}\left[\frac{2B_0^2}{|\nabla\Psi|^2}\kappa_\psi\xi_\psi+B_0^2\nabla\cdot\boldsymbol{\xi}+2\kappa_sB_0^2\xi_s+\frac{\gamma p_0}{\omega^2\rho_0}B_0^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla(\nabla\cdot\boldsymbol{\xi})}{B_0^2}\right)\right]=-\gamma p_0\nabla\cdot\boldsymbol{\xi} \tag{133}$$

Equations (130) and (133), which involves only surface derivative operators, can be put in the following matrix form:

$$\begin{pmatrix} E_{11} & E_{12}\\ E_{21} & E_{22}\end{pmatrix}\begin{pmatrix}\xi_s\\ \nabla\cdot\boldsymbol{\xi}\end{pmatrix}=\begin{pmatrix} F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}\begin{pmatrix}P_1\\ \xi_\psi\end{pmatrix} \tag{134}$$

with the matrix elements given by

$$E_{11}=-\frac{\omega^2\rho_0|\nabla\Psi|^2}{B_0^2}-\mu_0^{-1}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2}{B_0^2}\mathbf{B}_0\cdot\nabla\right) \tag{135}$$

$$E_{12}=-2\kappa_s\gamma p_0 \tag{136}$$

$$E_{21}=2\mu_0^{-1}\kappa_s \tag{137}$$

$$E_{22}=\frac{\mu_0^{-1}B_0^2+\gamma p_0}{B_0^2}+\mu_0^{-1}\frac{\gamma p_0}{\omega^2\rho_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right) \tag{138}$$

$$F_{11}=2\kappa_s-\left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right)\cdot\nabla \tag{139}$$

$$F_{12}=\sigma\mathbf{B}_0\cdot\nabla-\mu_0^{-1}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2}{B_0^2}S\right)+2\kappa_s\frac{dp_0}{d\Psi} \tag{140}$$

$$F_{21}=-\frac{1}{B_0^2} \tag{141}$$

$$F_{22}=-\mu_0^{-1}\frac{2}{|\nabla\Psi|^2}\kappa_\psi. \tag{142}$$

Equations (134)-(142) agree with equations (25), (28), and (29) of Ref [3].

Using Eq. (129) to eliminate Q_b in Eq. (127), we obtain

$$\begin{aligned}\nabla\Psi\cdot\nabla P_1 \;&=\; \omega^2\rho_0\xi_\psi+\mu_0^{-1}|\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla\xi_\psi}{|\nabla\Psi|^2}\right)\\ &+\; (\mu_0^{-1}|\nabla\Psi|^2S-B_0^2\sigma)\frac{|\nabla\Psi|^2}{B_0^2}(\mathbf{B}_0\cdot\nabla\xi_s-S\xi_\psi)\\ &+\; 2\kappa_\psi[P_1+\gamma p_0\nabla\cdot\boldsymbol{\xi}+\xi_\psi p_0']\end{aligned} \tag{143}$$

The equation for the divergence of $\boldsymbol{\xi}$ [Eq. (95)] is written

$$\nabla\Psi\cdot\nabla\xi_\psi=|\nabla\Psi|^2\nabla\cdot\boldsymbol{\xi}-|\nabla\Psi|^2\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)\xi_\psi-|\nabla\Psi|^2\frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2}\cdot\nabla\xi_s+2|\nabla\Psi|^2\kappa_s\xi_s+\frac{\gamma p_0|\nabla\Psi|^2}{\omega^2\rho_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla(\nabla\cdot\boldsymbol{\xi})}{B_0^2}\right). \tag{144}$$

Equations (143) and (144) can be put in the following matrix form

$$\nabla\Psi\cdot\nabla\begin{pmatrix}P_1\\ \xi_\psi\end{pmatrix}=\begin{pmatrix} C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}\begin{pmatrix}P_1\\ \xi_\psi\end{pmatrix}+\begin{pmatrix} D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}\begin{pmatrix}\xi_s\\ \nabla\cdot\boldsymbol{\xi}\end{pmatrix}, \tag{145}$$

with the matrix elements given by

$$C_{11}=2\kappa_\psi \tag{146}$$

$$C_{12}=\omega^2\rho_0+\mu_0^{-1}|\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{|\nabla\Psi|^2}\right)-(\mu_0^{-1}|\nabla\Psi|^2S-B_0^2\sigma)\frac{|\nabla\Psi|^2}{B_0^2}S+2\kappa_\psi\frac{dp_0}{d\Psi} \tag{147}$$

$$D_{11}=(\mu_0^{-1}|\nabla\Psi|^2S-B_0^2\sigma)\frac{|\nabla\Psi|^2}{B_0^2}\mathbf{B}_0\cdot\nabla \tag{148}$$

$$D_{12}=2\gamma p_0\kappa_\psi \tag{149}$$

$$C_{21}=0 \tag{150}$$

$$C_{22}=-|\nabla\Psi|^2\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) \tag{151}$$

$$D_{21}=-|\nabla\Psi|^2\frac{(\mathbf{B}_0\times\nabla\Psi)}{B_0^2}\cdot\nabla+2|\nabla\Psi|^2\kappa_s \tag{152}$$

$$D_{22}=|\nabla\Psi|^2+\frac{\gamma p_0|\nabla\Psi|^2}{\omega^2\rho_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right). \tag{153}$$

Equations (146)-(153) agree with Equations (26) and (27) in Ref.[3]. (It took me several months to manage to put the equations in the matrix form given here.)

4.9 Normalized form of eigenmodes equations

Denote the strength of the equilibrium magnetic field at the magnetic axis by B_a, the mass density at the magnetic axis by ρ_a, and the major radius of the magnetic axis by R_a. Define a characteristic speed $V_A \equiv B_a / \sqrt{\rho_a \mu_0}$, which is the Alfvén speed at the magnetic axis. Using the Alfvén speed, we define a characteristic frequency $\omega_A \equiv V_A / R_a$. Multiplying the matrix equation (134) by $2\mu_0 / B_a^2$ gives

$$\begin{pmatrix} \overline{E}_{11} & \overline{E}_{12} \\ \overline{E}_{21} & \overline{E}_{22} \end{pmatrix} \begin{pmatrix} \xi_s \\ \nabla \cdot \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} F_{11} & \overline{F}_{12} \\ F_{21} & \overline{F}_{22} \end{pmatrix} \begin{pmatrix} \overline{P}_1 \\ \xi_\psi \end{pmatrix}, \tag{154}$$

with new quantities defined as follows: $\overline{P}_1 \equiv P_1 / [B_a^2 / (2\mu_0)]$, $\overline{E}_{11} \equiv E_{11} \frac{2\mu_0}{B_a^2}$, $\overline{E}_{12} \equiv E_{12} \frac{2\mu_0}{B_a^2}$, $\overline{F}_{12} \equiv F_{12} \frac{2\mu_0}{B_a^2}$, $\overline{E}_{21} \equiv E_{21} \frac{2\mu_0}{B_a^2}$, $\overline{E}_{22} \equiv E_{22} \frac{2\mu_0}{B_a^2}$, and $\overline{F}_{22} \equiv F_{22} \frac{2\mu_0}{B_a^2}$. Using the equations (135), (136), (140), (137), (138), and (142), the expression of $\overline{E}_{11}$,$\overline{E}_{12}$, $\overline{E}_{21}$, $\overline{E}_{22}$,$\overline{F}_{12}$, and $\overline{E}_{22}$ are written respectively as

$$\overline{E}_{11} = -\frac{2\varpi^2 \overline{\rho}_0}{R_a^2} \frac{|\nabla\Psi|^2}{B_0^2} - \frac{2}{B_a^2} \mathbf{B}_0 \cdot \nabla \left(\frac{|\nabla\Psi|^2}{B_0^2} \mathbf{B}_0 \cdot \nabla \right), \tag{155}$$

$$\overline{E}_{12} = -2\kappa_s \gamma \overline{p}_0, \tag{156}$$

$$\overline{E}_{21} = \kappa_s \frac{4}{B_a^2}, \tag{157}$$

$$\overline{E}_{22} = \frac{2B_0^2 / B_a^2 + \gamma \overline{p}_0}{B_0^2} + \frac{R_a^2}{B_a^2} \frac{\gamma \overline{p}_0}{\varpi^2 \overline{\rho}_0} \mathbf{B}_0 \cdot \nabla \left(\frac{\mathbf{B}_0 \cdot \nabla}{B_0^2} \right), \tag{158}$$

$$\overline{F}_{12} = \frac{2}{B_a^2} \mu_0 \sigma \mathbf{B}_0 \cdot \nabla - \frac{2}{B_a^2} \mathbf{B}_0 \cdot \nabla \left(\frac{|\nabla\Psi|^2}{B_0^2} S \right) + 2\kappa_s \frac{d\overline{p}_0}{d\Psi}, \tag{159}$$

$$\overline{F}_{22} = -\frac{4}{B_a^2} \frac{\kappa_\psi}{|\nabla\Psi|^2}, \tag{160}$$

where $\varpi = \omega / \omega_A$, $\overline{\rho}_0 = \rho_0 / \rho_a$, $\overline{p}_0 = p_0 / [B_a^2 / (2\mu_0)]$. Next, consider re-normalizing the matrix equation (145). Multiply the first equation of matrix equation (145) by $2\mu_0 / B_a^2$, giving

$$\nabla\Psi \cdot \nabla \begin{pmatrix} \overline{P}_1 \\ \xi_\psi \end{pmatrix} = \begin{pmatrix} C_{11} & \overline{C}_{12} \\ 0 & C_{22} \end{pmatrix} \begin{pmatrix} \overline{P}_1 \\ \xi_\psi \end{pmatrix} + \begin{pmatrix} \overline{D}_{11} & \overline{D}_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \xi_s \\ \nabla \cdot \boldsymbol{\xi} \end{pmatrix}, \tag{161}$$

with the new matrix elements defined as follows: $\overline{C}_{12} \equiv C_{12} \frac{2\mu_0}{B_a^2}$, $\overline{D}_{11} \equiv D_{11} \frac{2\mu_0}{B_a^2}$, and $\overline{D}_{12} \equiv D_{12} \frac{2\mu_0}{B_a^2}$. (Note that, although the second equation of matrix equation (161) uses $\overline{P}_1$, instead of P_1, as a variable , it is actually identical with the second equation of matrix equation (145) because the $\overline{P}_1$ term is multiplied by zero.) Using Eqs. (147), (148) (149) , we obtain

$$\overline{C}_{12} = \frac{2}{R_a^2} \varpi^2 \overline{\rho}_0 + \frac{2}{B_a^2} |\nabla\Psi|^2 \mathbf{B}_0 \cdot \nabla \left(\frac{\mathbf{B}_0 \cdot \nabla}{|\nabla\Psi|^2} \right) - \frac{2}{B_a^2} (|\nabla\Psi|^2 S - B_0^2 \mu_0 \sigma) \frac{|\nabla\Psi|^2 S}{B_0^2} + 2\kappa_\psi \frac{d\overline{p}_0}{d\Psi}. \tag{162}$$

$$\overline{D}_{11} = \frac{2}{B_a^2} (|\nabla\Psi|^2 S - B_0^2 \mu_0 \sigma) \frac{|\nabla\Psi|^2}{B_0^2} \mathbf{B}_0 \cdot \nabla \tag{163}$$

$$\overline{D}_{12} = 2\gamma \overline{p}_0 \kappa_\psi. \tag{164}$$

Note that, after the normalization, all the coefficients of the resulting equations are of the order 10^0, thus, are suitable for accurate numerical calculation. Also note that, for typical tokamak plasmas, the normalization factor $2\mu_0 / B_a^2$ is of the order 10^{-6}, which is six order away from 10^0. Therefore the normalization performed here is necessary for accurate numerical calculation. [If the normalizing factor is two (or less) order from 10^0, then, from my experience, it is usually not necessary to perform additional normalization for the purpose of optimizing the numerical accuracy, i.e., the original units system has provided a reasonable normalization. Of course, suitable re-normalization will be of benefit to developing a clear physical insight into the problem in question.]

5 Flux coordinate system

In the above, we do not specify which coordinate system to use. In this article, we will use the straight-line magnetic surface coordinate system (i.e. flux coordinate system) (ψ, θ, ζ). The details of this coordinate system are given in my notes on tokamak equilibrium (/home/yj/theory/tokamak_equilibrium/tokamak_equilibrium.tm).

5.1 Magnetic field expression in flux coordinate system

Any axisymetrical magnetic field consistent with the equilibrium equation $\nabla p_0 = \mathbf{J}_0 \times \mathbf{B}_0$ can be written in the form

$$\mathbf{B}_0 = \nabla\Psi \times \nabla\phi + g(\Psi)\nabla\phi,$$

where $\Psi \equiv A_\phi R$. In the straight-line magnetic surface coordinates system (ψ, θ, ζ), the contra-variant form of the equilibrium magnetic field is expressed as

$$\mathbf{B}_0 = -\Psi'[\nabla\zeta \times \nabla\psi + q(\psi)\nabla\psi \times \nabla\theta], \tag{165}$$

where $\Psi' = d\Psi/d\psi$. The covariant form of the equilibrium magnetic field is given by

$$\mathbf{B}_0 = \left(\Psi'\frac{\mathcal{J}}{R^2}\nabla\psi \cdot \nabla\theta + gq\frac{\partial\delta}{\partial\psi} + g\delta q'\right)\nabla\psi + \left(-\frac{B_0^2}{\Psi'}\mathcal{J} - gq\right)\nabla\theta + g\nabla\zeta. \tag{166}$$

where $\mathcal{J}$ is the Jacobian of (ψ, θ, ζ) coordinate system.

5.2 Radial differential operator

The form of the radial differential operator $\nabla\Psi \cdot \nabla$ in (ψ, θ, ζ) coordinates is given by

$$\nabla\Psi \cdot \nabla = \Psi'|\nabla\psi|^2\frac{\partial}{\partial\psi} + \Psi'(\nabla\theta \cdot \nabla\psi)\frac{\partial}{\partial\theta} + \Psi'(\nabla\psi \cdot \nabla\zeta)\frac{\partial}{\partial\zeta}. \tag{167}$$

(Refer to my notes "tokamak_equilibrium.tm" for the proof.)

5.3 Surface differential operators

The MHD eigenmode equations (154) and (161) involve two surface operators, $\mathbf{B}_0 \cdot \nabla$ and $(\mathbf{B}_0 \times \nabla\Psi/B_0^2) \cdot \nabla$ (they are called surface operators because they involve only differential on magnetic surfaces). Next, we provide the form of the two operators in flux coordinate system (ψ, θ, ζ). Using Eq. (165), the $\mathbf{B}_0 \cdot \nabla$ operator (usually called magnetic differential operator) is written

$$\mathbf{B}_0 \cdot \nabla = -\Psi'\mathcal{J}^{-1}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right). \tag{168}$$

Using the covariant form of the equilibrium magnetic field [Eq. (166)], the $(\mathbf{B}_0 \times \nabla\Psi/B_0^2) \cdot \nabla$ operator is written

$$\frac{\mathbf{B}_0 \times \nabla\Psi}{B_0^2} \cdot \nabla = \left(1 + \Psi' g\frac{\mathcal{J}^{-1}}{B_0^2}q\right)\frac{\partial}{\partial\zeta} + \Psi' g\frac{\mathcal{J}^{-1}}{B_0^2}\frac{\partial}{\partial\theta}. \tag{169}$$

5.4 Fourier expansion over θ and ζ

A perturbation $G(\psi, \theta, \varphi)$ must be a periodic function of the poloidal angle θ and toroidal angle ζ, and thus can be expanded as the following two-dimensional Fourier series,

$$G(\psi, \theta, \zeta) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G_{nm}(\psi)e^{i(m\theta - n\zeta)}, \tag{170}$$

where the expansion coefficient G_{nm} is given by

$$G_{nm}(\psi) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} G(\psi, \theta, \zeta)e^{i(n\zeta - m\theta)}d\theta d\zeta. \tag{171}$$

Our next task is to derive the equations that the coefficients G_{nm} must satisfy.

5.5 Surface operator acting on perturbation

Next, consider the calculation of the surface operators acting on the above perturbation. Using Eq. (168), we obtain

$$\mathbf{B}_0 \cdot \nabla(aG) = -\Psi'\mathcal{J}^{-1}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\frac{\partial a}{\partial\theta} + i(m - nq)a\right]G_{nm}e^{i(m\theta - n\zeta)}, \tag{172}$$

where $a = a(\psi, \theta)$ is a known function that is independent of ζ. Similarly, we have

$$\begin{aligned}
&\mathbf{B}_0 \cdot \nabla(a\mathbf{B}_0 \cdot \nabla G) \\
= & -\Psi'\mathbf{B}_0 \cdot \nabla\left[a\mathcal{J}^{-1}\sum_{m,n} i(m - nq)G_{nm}(\psi)e^{i(m\theta - n\zeta)}\right] \\
= & (\Psi')^2\mathcal{J}^{-1}\sum_{m,n}\left[\frac{\partial}{\partial\theta}(a\mathcal{J}^{-1}) + i(m - nq)a\mathcal{J}^{-1}\right]i(m - nq)G_{nm}(\psi)e^{i(m\theta - n\zeta)} \\
= & (\Psi')^2\sum_{m,n}\left[i(m - nq)\mathcal{J}^{-1}\frac{\partial}{\partial\theta}(a\mathcal{J}^{-1}) - (m - nq)^2 a\mathcal{J}^{-2}\right]G_{nm}(\psi)e^{i(m\theta - n\zeta)}.
\end{aligned} \tag{173}$$

Using Eq. (169), we obtain

$$\begin{aligned}&\left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right)\cdot\nabla G\\ =&\left(1+\Psi' gq\frac{\mathcal{J}^{-1}}{B_0^2}\right)(-in)\sum_{m,n}G_{nm}(\psi)e^{i(m\theta-n\zeta)}+\Psi' g\frac{\mathcal{J}^{-1}}{B_0^2}\sum_{m,n}imG_{nm}(\psi)e^{i(m\theta-n\zeta)}\\ =&\sum_{m,n}\left[i(m-nq)\Psi' g\frac{\mathcal{J}^{-1}}{B_0^2}-in\right]G_{nm}(\psi)e^{i(m\theta-n\zeta)}.\end{aligned}\tag{174}$$

6 Discrete form of elements of matrix $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$, and $\boldsymbol{F}$

The elements of matrix C, D, E, and F are two dimensional differential operators about (θ,ζ), which can be called surface differential operators. As discussed above, we use Fourier expansion to treat the differential with respect to θ and ζ. In this method, we need to take inner product between different Fourier harmonics. Noting this, we recognize that it is useful to define the following inner product operator:

$$\langle W\rangle_{n'nm'm}\equiv\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\exp[i(n'\zeta-m'\theta)]W\exp[i(m\theta-n\zeta)]d\theta d\zeta,$$

where W is a surface differential operator of the following form

$$W=a(\psi,\theta)\frac{\partial}{\partial\theta}+b(\psi,\theta)\frac{\partial}{\partial\zeta}.\tag{175}$$

Because both of the coefficients in expression (175) are independent of ζ, it is ready to see that, for $n'\neq n$, $\langle W\rangle_{n'nm'm}=0$. This indicates that G_{nm} with different n are decoupled with each other.

For notation ease, $\langle W\rangle_{n'nm'm}$ is denoted by $\langle W\rangle_{m'm}$ when $n'=n$, i.e.,

$$\langle W\rangle_{m'm}=\frac{1}{2\pi}\int_0^{2\pi}\exp[i(n\zeta-m'\theta)]W\exp[i(m\theta-n\zeta)]d\theta,$$

[For the special case that W is an algebra operator $W=W(\psi,\theta)$, equation (175) can be reduced to

$$\langle W\rangle_{m'm}=\frac{1}{2\pi}\int_0^{2\pi}e^{i(m-m')\theta}Wd\theta,\tag{176}$$

where $W(\psi,\theta)$ is independent of ζ because, as we will see below, W is determined by equilibrium quantities (for example, $W(\psi,\theta)=\frac{\mathcal{J}^{-2}}{B_0^2}$), which is axisymetrical. Expression (176) is a Fourier integration over the interval $[0,2\pi]$, which can be efficiently calculated by using the FFT algorithm (details are given in Chapter 13.9 of Ref. [5]).] After using the Fourier harmonics expansion and taking the inner product, every element of the matrices C, D, E, and F becomes a $L\times L$ matrix, where L is the total number of the Fourier harmonics included in the expansion. Taking the matrix $\overline{E}$ as an example, it is discretized as

$$\begin{pmatrix}\overline{E}_{11} & \overline{E}_{12}\\ \overline{E}_{21} & \overline{E}_{22}\end{pmatrix}\rightarrow\begin{pmatrix}\overline{E}_{11}^{(11)} & \ldots & \overline{E}_{11}^{(1L)} & \overline{E}_{12}^{(11)} & \ldots & \overline{E}_{12}^{(1L)}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \overline{E}_{11}^{(L1)} & \ldots & \overline{E}_{11}^{(LL)} & \overline{E}_{12}^{(L1)} & \ldots & \overline{E}_{12}^{(LL)}\\ \overline{E}_{21}^{(11)} & \ldots & \overline{E}_{21}^{(1L)} & \overline{E}_{22}^{(11)} & \ldots & \overline{E}_{22}^{(1L)}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \overline{E}_{21}^{(L1)} & \ldots & \overline{E}_{21}^{(LL)} & \overline{E}_{22}^{(L1)} & \ldots & \overline{E}_{22}^{(LL)}\end{pmatrix},\tag{177}$$

where $\overline{E}_{11}^{(m'm)}\equiv\langle\overline{E}_{11}\rangle_{m'm}$, $\overline{E}_{12}^{(m'm)}\equiv\langle\overline{E}_{12}\rangle_{m'm}$, $\overline{E}_{21}^{(m'm)}\equiv\langle\overline{E}_{21}\rangle_{m'm}$, $\overline{E}_{22}^{(m'm)}\equiv\langle\overline{E}_{22}\rangle_{m'm}$. Next, let us derive the expressions of $\overline{E}_{11}^{(m'm)}$, $\overline{E}_{12}^{(m'm)}$, $\overline{E}_{21}^{(m'm)}$, $\overline{E}_{22}^{(m'm)}$. The goal of he derivation is to perform the surface differential operators so that all the inner products take the form of the Fourier integration given by Eq. (176). For the convenience of reference, the expression of matrix $\overline{E}$ is repeated here:

$$\overline{E}=\begin{pmatrix}-\frac{2\varpi^2\overline{p}_0}{R_a^2}\frac{|\nabla\Psi|^2}{B_0^2}-\frac{2}{B_a^2}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2}{B_0^2}\mathbf{B}_0\cdot\nabla\right) & -2\kappa_s\gamma\overline{p}_0\\ \kappa_s\frac{4}{B_a^2} & \frac{2B_0^2/B_a^2+\gamma\overline{p}_0}{B_0^2}+\frac{R_a^2}{B_a^2}\frac{\gamma\overline{p}_0}{\varpi^2\overline{\rho}_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right)\end{pmatrix}.$$

Then $\overline{E}_{11}^{(m'm)}$ is written as

$$\begin{aligned}\overline{E}_{11}^{m'm}&\equiv\langle\overline{E}_{11}\rangle_{m'm}\\ &=-\frac{2\varpi^2\overline{p}_0}{R_a^2}\left\langle\frac{|\nabla\Psi|^2}{B_0^2}\right\rangle_{m'm}-\frac{2}{B_a^2}\left\langle\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2}{B_0^2}\mathbf{B}_0\cdot\nabla\right)\right\rangle_{m'm}\end{aligned}\tag{178}$$

Making use of Eq. (173), equation (178) is written as

$$\overline{E}_{11}^{m'm} = -\frac{2\varpi^2\overline{\rho}_0}{R_a^2}\left\langle \frac{|\nabla\Psi|^2}{B_0^2}\right\rangle_{m'm} - \frac{2}{B_a^2}(\Psi')^2\left[i(m - n\,q)\left\langle \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2\mathcal{J}^{-1}}{B_0^2}\right)\right\rangle_{m'm} - (m - n\,q)^2\left\langle\frac{|\nabla\Psi|^2\mathcal{J}^{-2}}{B_0^2}\right\rangle_{m'm}\right]. \tag{179}$$

Note that all the operators within the inner operator $\langle ...\rangle_{m'm}$ of the above equation are algebra operators. Therefore the calculation of the inner product $\langle ...\rangle_{m'm}$ reduces to the calculation of the Fourier integration (176), which can be efficiently calculated by using the FFT algorithm (it is thus implemented in GTAW). Similarly, the discrete form of the other matrix elements are written respectively as:

$$\begin{aligned}\overline{E}_{12}^{m'm} &\equiv \langle \overline{E}_{12}\rangle_{m'm}\\ &= -2\gamma\overline{p}_0\langle\kappa_s\rangle_{m'm}.\end{aligned} \tag{180}$$

$$\begin{aligned}\overline{E}_{21}^{m'm} &\equiv \langle \overline{E}_{21}\rangle_{m'm}\\ &= \frac{4}{B_a^2}\langle\kappa_s\rangle_{m'm}\end{aligned} \tag{181}$$

$$\begin{aligned}\overline{E}_{22}^{m'm} &= \langle \overline{E}_{22}\rangle_{m'm}\\ &= \frac{2}{B_a^2}\langle 1\rangle_{m'm} + \gamma\overline{p}_0\left\langle\frac{1}{B_0^2}\right\rangle + \frac{R_a^2}{B_a^2}\frac{\gamma\overline{p}_0}{\varpi^2\overline{\rho}_0}\left\langle \mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right)\right\rangle_{m'm}\\ &= \frac{2}{B_a^2}\langle 1\rangle_{m'm} + \gamma\overline{p}_0\left\langle\frac{1}{B_0^2}\right\rangle + \frac{R_a^2}{B_a^2}\frac{\gamma\overline{p}_0}{\varpi^2\overline{\rho}_0}(\Psi')^2\left[i(m - n\,q)\left\langle\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0^2}\right)\right\rangle_{m'm} - (m - n\,q)^2\left\langle\frac{\mathcal{J}^{-2}}{B_0^2}\right\rangle_{m'm}\right]\end{aligned} \tag{182}$$

Next, consider the discrete form of the normalized F matrix, which is given by

$$\begin{pmatrix} F_{11} & \overline{F}_{12}\\ F_{21} & \overline{F}_{22}\end{pmatrix} = \begin{pmatrix} 2\kappa_s - \left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right)\cdot\nabla, & \frac{2}{B_a^2}\mu_0\sigma\mathbf{B}_0\cdot\nabla - \frac{2}{B_a^2}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2 S}{B_0^2}\right) + 2\kappa_s\frac{d\,\overline{p}_0}{d\Psi}\\ -\frac{1}{B_0^2}, & -\frac{4}{B_a^2}\frac{\kappa_\psi}{|\nabla\Psi|^2}\end{pmatrix} \tag{183}$$

Using Eqs. (139) and (159), we obtain

$$\begin{aligned}F_{11}^{m'm} &= \langle F_{11}\rangle_{m'm}\\ &= 2\langle\kappa_s\rangle_{m'm} - \left\langle\left(\frac{\mathbf{B}_0\times\nabla\Psi}{B_0^2}\right)\cdot\nabla\right\rangle_{m'm}\\ &= 2\langle\kappa_s\rangle_{m'm} - i(m - n\,q)\Psi' g\left\langle\frac{\mathcal{J}^{-1}}{B_0^2}\right\rangle_{m'm} + i\,n\langle 1\rangle_{m'm}\end{aligned} \tag{184}$$

$$\begin{aligned}\overline{F}_{12}^{m'm} &= \langle\overline{F}_{12}\rangle_{m'm}\\ &= \left\langle\frac{2}{B_a^2}\mu_0\sigma\mathbf{B}_0\cdot\nabla - \frac{2}{B_a^2}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2 S}{B_0^2}\right) + 2\kappa_s\frac{d\,\overline{p}_0}{d\Psi}\right\rangle_{m'm}\\ &= \frac{2}{B_a^2}\langle\mu_0\sigma\mathbf{B}_0\cdot\nabla\rangle_{m'm} - \frac{2}{B_a^2}\left\langle\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2 S}{B_0^2}\right)\right\rangle_{m'm} + 2\frac{d\,\overline{p}_0}{d\Psi}\langle\kappa_s\rangle_{m'm}\\ &= -\Psi'\frac{2}{B_a^2}i(m - n\,q)\langle\mu_0\sigma\mathcal{J}^{-1}\rangle_{m'm} + \Psi'\frac{2}{B_a^2}\left[i(m - n\,q)\left\langle\frac{|\nabla\Psi|^2 S}{B_0^2}\mathcal{J}^{-1}\right\rangle_{m'm} + \left\langle\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2 S}{B_0^2}\right)\right\rangle_{m'm}\right] + 2\frac{d\,\overline{p}_0}{d\Psi}\langle\kappa_s\rangle_{m'm}\end{aligned} \tag{185}$$

$$\langle F_{21}\rangle_{m'm} = -\left\langle\frac{1}{B_0^2}\right\rangle_{m'm} \tag{186}$$

$$\langle\overline{F}_{22}\rangle_{m'm} = -\frac{4}{B_a^2}\left\langle\frac{\kappa_\psi}{|\nabla\Psi|^2}\right\rangle_{m'm}, \tag{187}$$

Next, we derive the discrete form of matrix C and D. Before doing this, we examine matrix equation (161), which can be written as

$$\begin{pmatrix}\nabla\Psi\cdot\nabla & 0\\ 0 & \nabla\Psi\cdot\nabla\end{pmatrix}\begin{pmatrix}\overline{P}_1\\ \xi_\psi\end{pmatrix} = \begin{pmatrix} C_{11} & \overline{C}_{12}\\ 0 & C_{22}\end{pmatrix}\begin{pmatrix}\overline{P}_1\\ \xi_\psi\end{pmatrix} + \begin{pmatrix}\overline{D}_{11} & \overline{D}_{12}\\ D_{21} & D_{22}\end{pmatrix}\begin{pmatrix}\xi_s\\ \nabla\cdot\boldsymbol{\xi}\end{pmatrix}. \tag{188}$$

Using the expression of the operator $\nabla\Psi\cdot\nabla$, i.e.,

$$\nabla\Psi\cdot\nabla = \Psi'|\nabla\psi|^2\frac{\partial}{\partial\psi} + \Psi'(\nabla\theta\cdot\nabla\psi)\frac{\partial}{\partial\theta} + \Psi'(\nabla\zeta\cdot\nabla\psi)\frac{\partial}{\partial\zeta}, \tag{189}$$

equation (188) is written as

$$\begin{pmatrix} \Psi'|\nabla\psi|^2 & 0 \\ 0 & \Psi'|\nabla\psi|^2 \end{pmatrix}\begin{pmatrix} \frac{\partial \overline{P}_1}{\partial\psi} \\ \frac{\partial \xi_\psi}{\partial\psi} \end{pmatrix} = \begin{pmatrix} -\Psi'\nabla\theta\cdot\nabla\psi\frac{\partial}{\partial\theta} - \Psi'(\nabla\zeta\cdot\nabla\psi)\frac{\partial}{\partial\zeta} & 0 \\ 0 & -\Psi'\nabla\theta\cdot\nabla\psi\frac{\partial}{\partial\theta} - \Psi'(\nabla\zeta\cdot\nabla\psi)\frac{\partial}{\partial\zeta} \end{pmatrix}\begin{pmatrix} \overline{P}_1 \\ \xi_\psi \end{pmatrix} +$$

$$\begin{pmatrix} C_{11} & \overline{C}_{12} \\ 0 & C_{22} \end{pmatrix}\begin{pmatrix} \overline{P}_1 \\ \xi_\psi \end{pmatrix} + \begin{pmatrix} \overline{D}_{11} & \overline{D}_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} \xi_s \\ \nabla\cdot\boldsymbol{\xi} \end{pmatrix} \tag{190}$$

Define the first matrix on the r.h.s of the above equation as $\mathcal{H}$, then $\mathcal{H}_{12} = \mathcal{H}_{21} = 0$, and $\mathcal{H}_{11}$ and $\mathcal{H}_{22}$ are given by

$$\mathcal{H}_{11} = \mathcal{H}_{22} = -\Psi'\nabla\theta\cdot\nabla\psi\frac{\partial}{\partial\theta} - \Psi'(\nabla\zeta\cdot\nabla\psi)\frac{\partial}{\partial\zeta}. \tag{191}$$

Then

$$\langle\mathcal{H}_{11}\rangle_{m'm} = \langle\mathcal{H}_{22}\rangle_{m'm} = -im\Psi'\langle\nabla\theta\cdot\nabla\psi\rangle_{m'm} + in\Psi'\langle\nabla\zeta\cdot\nabla\psi\rangle_{m'm} \tag{192}$$

$$\langle C_{11}\rangle_{m'm} = 2\langle\kappa_\psi\rangle_{m'm} \tag{193}$$

$$\begin{aligned}\langle\overline{C}_{12}\rangle &= \frac{2}{R_a^2}\varpi^2\overline{\rho}_0\langle 1\rangle_{m'm} + \frac{2}{B_a^2}\left\langle |\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{|\nabla\Psi|^2}\right)\right\rangle_{m'm} - \frac{2}{B_a^2}\left\langle (|\nabla\Psi|^2 S - B_0^2\mu_0\sigma)\frac{|\nabla\Psi|^2}{B_0^2}S\right\rangle_{m'm} + 2\langle\kappa_\psi\rangle_{m'm}\frac{d\overline{p}_0}{d\Psi} \\ &= \frac{2}{R_a^2}\varpi^2\overline{\rho}_0\langle 1\rangle_{m'm} + \frac{2}{B_a^2}(\Psi')^2\left[i(m - nq)\left\langle |\nabla\Psi|^2\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{|\nabla\Psi|^2}\right)\right\rangle_{m'm} - (m - nq)^2\langle\mathcal{J}^{-2}\rangle_{m'm}\right] - \\ &\quad \frac{2}{B_a^2}\left\langle (|\nabla\Psi|^2 S - B_0^2\mu_0\sigma)\frac{|\nabla\Psi|^2 S}{B_0^2}\right\rangle_{m'm} + 2\langle\kappa_\psi\rangle_{m'm}\frac{d\overline{p}_0}{d\Psi}\end{aligned}$$

$$\langle C_{22}\rangle_{m'm} = -\left\langle |\nabla\Psi|^2\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)\right\rangle_{m'm} \tag{194}$$

The formula for calculating the right-hand side of Eq. (194) is given in Sec. 9.6.

$$\langle\overline{D}_{11}\rangle_{m'm} = -i(m - nq)\frac{2}{B_a^2}\Psi'\left\langle (|\nabla\Psi|^2 S - B_0^2\mu_0\sigma)\frac{\mathcal{J}^{-1}|\nabla\Psi|^2}{B_0^2}\right\rangle_{m'm} \tag{195}$$

$$\langle\overline{D}_{12}\rangle_{m'm} = 2\gamma\overline{p}_0\langle\kappa_\psi\rangle_{m'm}. \tag{196}$$

$$\langle D_{21}\rangle_{m'm} = -i(m - nq)\Psi' g\left\langle\frac{|\nabla\Psi|^2\mathcal{J}^{-1}}{B_0^2}\right\rangle_{m'm} + in\langle|\nabla\Psi|^2\rangle_{m'm} + 2\langle|\nabla\Psi|^2\kappa_s\rangle_{m'm} \tag{197}$$

$$\begin{aligned}\langle D_{22}\rangle_{m'm} &= \langle|\nabla\Psi|^2\rangle_{m'm} + \frac{\gamma\overline{p}_0}{\varpi^2\overline{\rho}_0}\frac{R_a^2}{2}\left\langle |\nabla\Psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right)\right\rangle_{m'm} \\ &= \langle|\nabla\Psi|^2\rangle_{m'm} + (\Psi')^2\frac{\gamma\overline{p}_0}{\varpi^2\overline{\rho}_0}\frac{R_a^2}{2}\left[i(m - nq)\left\langle |\nabla\Psi|^2\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0^2}\right)\right\rangle_{m'm} - (m - \right. \\ &\quad \left. nq)^2\left\langle\frac{|\nabla\Psi|^2\mathcal{J}^{-2}}{B_0^2}\right\rangle_{m'm}\right]\end{aligned} \tag{198}$$

6.1 Weight functions used in Fourier integration

In the GTAW code, the weight functions appearing in the Fourier integration are numbered as follows:

$$W_1(\psi,\theta) \equiv \frac{\mathcal{J}^{-2}}{B_0^2}, W_2(\psi,\theta) \equiv \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0^2}\right) \tag{199}$$

$$W_3 \equiv \frac{|\nabla\Psi|^2\mathcal{J}^{-2}}{B_0^2}, W_4 \equiv \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2\mathcal{J}^{-1}}{B_0^2}\right) \tag{200}$$

$$W_5 \equiv \frac{|\nabla\Psi|^2}{B_0^2}, W_6 \equiv \kappa_s \tag{201}$$

$$W_7 \equiv \frac{1}{B_0^2}, W_8 = \frac{\mathcal{J}^{-1}}{B_0^2}, W_9 = \mu_0\sigma\mathcal{J}^{-1} \tag{202}$$

$$W_{10} = \mathcal{J}^{-1}\frac{|\nabla\Psi|^2 S}{B_0^2}, W_{11} = \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2 S}{B_0^2}\right) \tag{203}$$

$$W_{12} = \frac{\kappa_\psi}{|\nabla\Psi|^2}, W_{13} = |\nabla\Psi|^2 \tag{204}$$

$$W_{14} = |\nabla\Psi|^2\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0^2}\right), W_{15} = \kappa_\psi \tag{205}$$

$$W_{16} = \frac{\Psi_R}{R} + \Psi_{RR} + \Psi_{ZZ} - \frac{1}{|\nabla\Psi|^2}(2\Psi_R\Psi_R\Psi_{RR} + 4\Psi_R\Psi_Z\Psi_{ZR} + 2\Psi_Z\Psi_Z\Psi_{ZZ}) \quad (206)$$

$$W_{17} = (|\nabla\Psi|^2 S - B_0^2\mu_0\sigma)\frac{|\nabla\Psi|^2 S}{B_0^2} \quad (207)$$

$$W_{18} = (|\nabla\Psi|^2 S - B_0^2\mu_0\sigma)\frac{|\nabla\Psi|^2 \mathcal{J}^{-1}}{B_0^2} \quad (208)$$

$$W_{19} = \nabla\theta\cdot\nabla\psi, W_{20} = \nabla\zeta\cdot\nabla\psi \quad (209)$$

$$W_{21} = |\nabla\Psi|^2\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{|\nabla\Psi|^2}\right), W_{22} = \mathcal{J}^{-2} \quad (210)$$

$$W_{23} = \frac{|\nabla\Psi|^2\mathcal{J}^{-1}}{B_0^2}, W_{24} = |\nabla\Psi|^2\kappa_s \quad (211)$$

The formulas for calculating the equilibrium quantities, such as the geodesic curvature κ_s, normal curvature κ_ψ, and the local magnetic shear S, are given in Sec. 9.4.

6.2 Numerical methods for finding continua

In the GTAW code, the matrix elements $\overline{E}_{21}^{m'm}$ and $\overline{E}_{22}^{m'm}$ are multiplied by ϖ^2 (check whether this will make $\varpi^2 = 0$ a root of $\text{Det}(\overline{E}(\varpi)) = 0$?). After this, the matrix elements $\overline{E}$ can be written in the following form

$$\overline{E} = \overline{E}_a + \varpi^2\overline{E}_b, \quad (212)$$

where $\overline{E}_a$ and $\overline{E}_b$ are $2L\times 2L$ matrix which are both independent of ϖ.

The continua are determined by the condition that $\text{Det}(\overline{E}(\varpi)) = 0$, which is the condition that the matrix equation $\overline{E}X = 0$ has nonzero solutions. Using Eq. (212), the matrix equation $\overline{E}X = 0$ can be written

$$\overline{E}_a X = -\varpi^2\overline{E}_b X. \quad (213)$$

Thus finding ϖ that can make $\overline{E}X = 0$ have nonzero solution reduces to finding the eigenvalues of the generalized eigenvalue problem in Eq. (213). In my code, the generalized eigenvalue problem in Eq. (213) is solved numerically by using the **zggev** subroutine in Lapack library. The numerical results of the continuous spectrum are given in Sec. 9.2.

6.3 Analytical approximations to continuous spectrum

Before we give the numerical solution for the continuous spectrum, we consider some approximations that can be made when solving continuous spectrum.

6.3.1 Slow sound approximation

Examining the expression (138) for matrix element E_{22}, we find that the first term of E_{22} can be written as

$$\frac{\mu_0^{-1}B_0^2 + \gamma p_0}{B_0^2} = \mu_0^{-1}\left(1 + \frac{\gamma\beta}{2}\right), \quad (214)$$

where $\beta \equiv p_0/(B_0^2/2\mu_0)$, while the second term of E_{22} can be written as

$$\begin{aligned} \mu_0^{-1}\frac{\gamma p_0}{\omega^2\rho_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right) &\approx \mu_0^{-1}\left(\frac{\gamma p_0}{\omega^2\rho_0}k_\parallel^2\right), \\ &\approx \mu_0^{-1}\left(\frac{\gamma p_0}{V_A^2\rho_0}\right) \\ &= \mu_0^{-1}\left(\frac{\gamma p_0}{\frac{B_0^2}{\mu_0\rho_0}\rho_0}\right) \\ &= \mu_0^{-1}\left(\frac{\gamma\beta}{2}\right). \end{aligned} \quad (215)$$

where $k_\parallel$ is the parallel wave vector and we have used the approximation $\omega/k_\parallel \approx V_A$. Using Eqs. (214) and (215), the ratio of the second term to the first term of E_{22} is written as $(\gamma\beta/2)/[1+\gamma\beta/2]$. For low β ($\beta \ll 1$) equilibrium, the ratio is small and therefore the second term of E_{22} can be dropped. This approximation is called the slow sound approximation in the literature[6, 7]. Numerical results indicate this approximation will remove all the sound continua while keeping the Alfven continua nearly unchanged.

6.3.2 Zero beta limit

If we set all the thermal pressure terms in E_{12} and E_{22} to be zero (this is equivalent with setting $\gamma = 0$), the sound wave will be removed from the system. This approximation is called zero beta limit in literature[6]. Numerical results indicate the zero beta limit will remove all the sound continua. Compared with the slow sound approximation, the zero beta limit will make the frequency of the Alfven continua a little lower, and make the zeroth continua gaps (BAE gaps) disappear[7].

6.3.3 Cylindrical geometry limit

Consider the form of matrix E in the cylindrical geometry limit, in which the equilibrium quantities are independent of poloidal angle. Equation (284) indicates that the geodesic curvature κ_s is zero in this case. Thus, the matrix elements E_{12} and E_{21} are zero. Next, consider the matrix elements E_{11} and E_{22}. Because all equilibrium quantities are independent of the poloidal angle, different poloidal harmonics of the perturbation are decoupled. Therefore, we can consider a perturbation with a single poloidal mode number . For a poloidal harmonic with poloidal mode number m, matrix element E_{11} is written

$$\begin{aligned} E_{11} &= -\frac{\omega^2\rho_0|\nabla\Psi|^2}{B_0^2} - \mu_0^{-1}\mathbf{B}_0\cdot\nabla\left(\frac{|\nabla\Psi|^2}{B_0^2}\mathbf{B}_0\cdot\nabla\right) \\ &= -\frac{\omega^2\rho_0|\nabla\Psi|^2}{B_0^2} + \mu_0^{-1}\frac{|\nabla\Psi|^2}{B_0^2}(\Psi'\mathcal{J}^{-1})^2(m-nq)^2, \end{aligned} \tag{216}$$

and matrix element E_{22} is written

$$\begin{aligned} E_{22} &= \frac{\mu_0^{-1}B_0^2+\gamma p_0}{B_0^2} + \mu_0^{-1}\frac{\gamma p_0}{\omega^2\rho_0}\mathbf{B}_0\cdot\nabla\left(\frac{\mathbf{B}_0\cdot\nabla}{B_0^2}\right) \\ &= \frac{\mu_0^{-1}B_0^2+\gamma p_0}{B_0^2} - \mu_0^{-1}\frac{\gamma p_0}{\omega^2\rho_0}\frac{1}{B_0^2}(\Psi'\mathcal{J}^{-1})^2(m-nq)^2. \end{aligned} \tag{217}$$

The continua are the roots of the equation $\det(E) = 0$, which, in the cylindrical geometry limit, reduces to

$$E_{11}E_{22} = 0. \tag{218}$$

Two branches of the roots of Eq. (218) are given by $E_{11} = 0$ and $E_{22} = 0$, respectively. The equation $E_{11} = 0$ is written

$$-\omega^2\rho_0 + \mu_0^{-1}(\Psi'\mathcal{J}^{-1})^2(m-nq)^2 = 0, \tag{219}$$

which gives

$$\omega^2 = \frac{(\Psi'\mathcal{J}^{-1})^2(m-nq)^2}{\mu_0\rho_0}, \tag{220}$$

which is the Alfvén branch of the continua. Figure 1a plots the results of Eq. (220). The equation $E_{22} = 0$ is written

$$\omega^2 = \frac{\gamma p_0}{\mu_0^{-1}B_0^2+\gamma p_0}\frac{(\Psi'\mathcal{J}^{-1})^2}{\mu_0\rho_0}(m-nq)^2, \tag{221}$$

which is the sound branch of the continua. Figure 1b plots the results of Eq. (221).

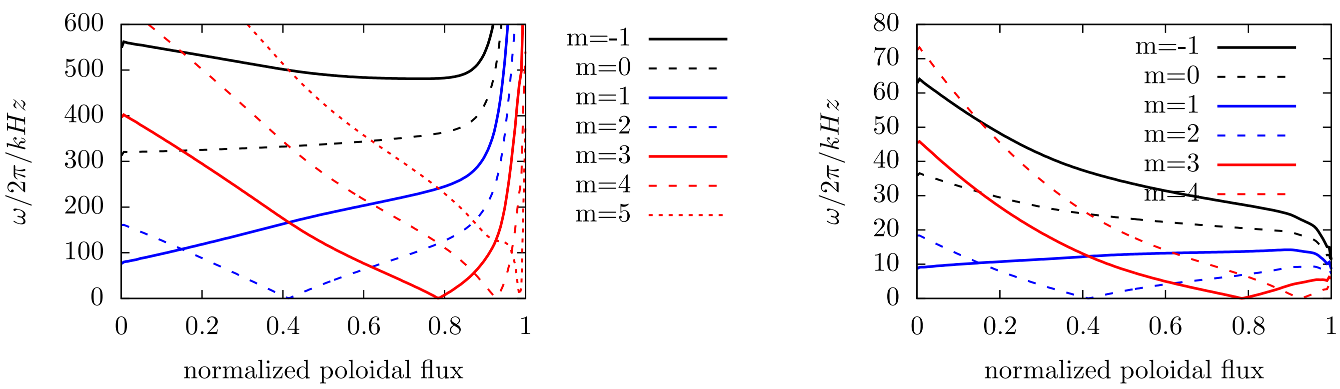

Figure 1. $n = 1$ Alfven continua (left) and sound continua (right) in the cylindrical geometry limit for $m = 0, 1, 2, 3, 4$, and 5 (calculated by using Eqs. (220) and (221)). The equilibrium used for this calculation is for EAST discharge #38300@3.9s (G-eqdsk filename g038300.03900, which was provided by Dr. Guoqiang Li). The number density of ions is given in Fig. 12. Because the Jacobian $\mathcal{J}$ in toroidal geometry depends on the poloidal angle, the average value of $\mathcal{J}$ on a magnetic surface is used in evaluating the right-hand side of (220).

Figures compares the Alfven continua in the cylindrical limit with those in the toroidal geometry. The results indicate that the Alfven continua in the toroidal geometry reconnect, forming gaps near the locations where the Alfven continua in the cylindrical limit intersect each other.

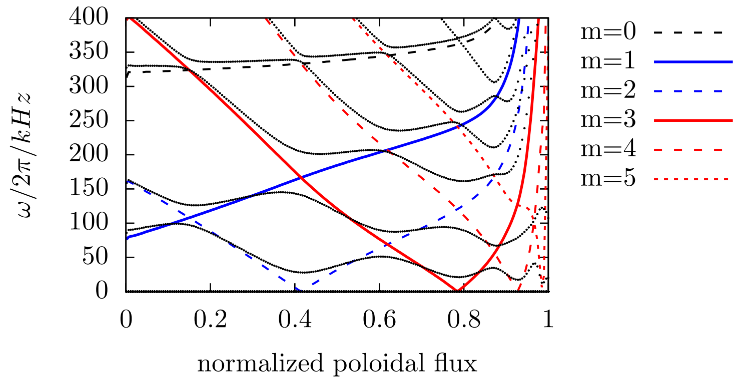

Figure 2. Comparision of the the Alfven continua in toroidal geometry (black solid lines) and Alfven continua in the cylindrical limit (other lines). The Alfven continua in toroidal geometry are obtained by using the slow-sound-approximation. The equilibrium is EAST discharge #38300 at 3.9s.

The result in Eq. (220) is not clear from the view of physics since it involves the Jacobian, which is a mathematical factor due to the freedom in the choice of coordinates. Next, we try to write the right-hand side of Eq. (220) in more physical form. In cylindrical geometry limit, magnetic surfaces are circular. Thus the radial coordinate can be chosen to be the geometrical radius of the circular magnetic surface, and the usual poloidal angle (i.e., equal-arc angle) can be used as the poloidal coordinate. Then the poloidal magnetic flux is written as

$$\Psi_p = \int_0^r B_p(r) 2\pi R_0 \, dr, \tag{222}$$

where $2\pi R_0$ is the length of the cylinder. We know that Ψ used in the Grad-Shafranov equation is related to Ψ_p by

$$\Psi = \pm \frac{\Psi_p}{2\pi} + C. \tag{223}$$

Using Eqs. (222) and (223), we obtain

$$\Psi' \equiv \frac{d\Psi}{dr} = \pm B_p R_0 \, . \tag{224}$$

Next, we calculate the Jacobian $\mathcal{J}$, which is defined by

$$\mathcal{J}^{-1} = \nabla\psi \times \nabla\theta \cdot \nabla\zeta. \tag{225}$$

Since we choose $\psi = r$ and $\zeta = \phi$ (the positive direction of θ is count clockwise when observers view along the positive direction of ϕ), the above equation is written

$$\begin{aligned} \mathcal{J}^{-1} &= \nabla r \times \nabla\theta \cdot \frac{\hat{\boldsymbol{\phi}}}{R_0}. \\ &= -\frac{1}{r R_0}. \end{aligned} \tag{226}$$

Using Eqs. (224) and (226) , $(\Psi' \mathcal{J}^{-1})^2$ is written

$$(\Psi' \mathcal{J}^{-1})^2 = \left(B_p R_0 \frac{1}{r R_0} \right)^2 = \frac{B_p^2}{r^2} \tag{227}$$

Using these, Eq. (220) is written

$$\omega^2 = \frac{B_p^2 (m - n q)^2}{r^2 \mu_0 \rho_0}. \tag{228}$$

Using the definition of safety factor in the cylindrical geometry

$$q = \frac{B_\phi r}{R_0 B_p}, \tag{229}$$

equation (228) is written

$$\omega^2 = \frac{B_\phi^2 (m - n q)^2}{\mu_0 \rho_0 q^2 R_0^2}. \tag{230}$$

In the cylindrical geometry, the parallel (to equilibrium magnetic field) wave-number is given by

$$k_\parallel = \frac{m - n q}{q R_0}. \tag{231}$$

Using this, Eq. (230) is written

$$\omega^2 = k_\parallel^2 \frac{B_\phi^2}{\mu_0 \rho_0}. \tag{232}$$

Using the definition of Alfven speed $V_{A\phi}^2 \equiv B_\phi^2 / (\mu_0 \rho_0)$, the above equation is written as

$$\omega^2 = k_\parallel^2 V_{A\phi}^2, \tag{233}$$

which gives the well known Alfven resonance condition. For later use, define

$$\omega_a^2 = \frac{B_p^2 (m - nq)^2}{r^2 \mu_0 \rho_0}, \tag{234}$$

then Eq. (228) is written as $\omega^2 = \omega_a^2$.

Similarly, by using Eq. (227), equation (217) for E_{22} is written as

$$E_{22} = \frac{\mu_0^{-1} B_0^2 + \gamma p_0}{B_0^2} - \mu_0^{-1} \frac{\gamma p_0}{\omega^2 \rho_0} \frac{1}{B_0^2} \frac{B_p^2}{r^2} (m - nq)^2.$$

Then equation $E_{22} = 0$ reduces to

$$\frac{\mu_0^{-1} B_0^2 + \gamma p_0}{B_0^2} - \mu_0^{-1} \frac{\gamma p_0}{\omega^2 \rho_0} \frac{1}{B_0^2} \frac{B_p^2}{r^2} (m - nq)^2 = 0 \tag{235}$$

$$\Rightarrow \omega^2 = \frac{\mu_0^{-1} \frac{\gamma p_0}{\rho_0} \frac{1}{B_0^2} \frac{B_p^2}{r^2} (m - nq)^2}{\frac{\mu_0^{-1} B_0^2 + \gamma p_0}{B_0^2}}$$

$$\Rightarrow \omega^2 = \frac{\mu_0^{-1} \frac{\gamma p_0}{\rho_0}}{\mu_0^{-1} B_0^2 + \gamma p_0} \frac{B_p^2}{r^2} (m - nq)^2$$

$$\Rightarrow \omega^2 = \frac{\frac{\gamma p_0}{\rho_0}}{\frac{B_0^2}{\mu_0 \rho_0} + \frac{\gamma p_0}{\rho_0}} \frac{B_p^2}{\mu_0 \rho_0 r^2} (m - nq)^2$$

$$\Rightarrow \omega^2 = \frac{C_s^2}{V_A^2 + C_s^2} \omega_a^2, \tag{236}$$

where $C_s^2 = \gamma p_0 / \rho_0$, $V_A^2 = B_0^2 / (\mu_0 \rho_0)$. Equation (166) gives the sound branch of the continua. For present tokamak plasma parameters, C_s is usually one order smaller than V_A. Thus, equation (236) indicates the sound continua are much smaller than the Alfven continua for the same m and n.

6.3.4 Approximate central frequency and radial location of continua gap

In the cylindrical geometry, continua with different poloidal mode numbers will intersect each other, as shown in Fig. 1. Next, we calculate the radial location of the intersecting point of two continua with poloidal mode number m and $m+1$, respectively. In the intersecting point, we have

$$\omega_{Am}^2 = \omega_{Am+1}^2, \tag{237}$$

i.e.

$$k_{\parallel m}^2 V_A^2 = k_{\parallel m+1}^2 V_A^2 \tag{238}$$

which gives

$$k_{\parallel m} = k_{\parallel m+1}, \tag{239}$$

or

$$k_{\parallel m} = -k_{\parallel m+1} \tag{240}$$

Inspecting the expression for $k_\parallel$ in Eq. (323), we know that only the case in Eq. (240) is possible, which gives

$$\frac{m - nq}{qR_0} = -\frac{(m+1) - nq}{qR_0}, \tag{241}$$

which further reduces to

$$q = \frac{2m+1}{2n} \equiv q_{\text{gap}}. \tag{242}$$

The above equation determines the radial location where the m continuum intersect the $m+1$ continuum. Note that a mode with two poloidal modes has two corresponding resonant surfaces. For the case where the mode has m and $m+1$ poloidal harmonics, the resonant surfaces are respectively $q = m / n$ and $q = (m+1) / n$. Note that the value of q given in Eq. (242) is between the above two values.

In toroidal geometry, the different poloidal modes are coupled, and the continuum will "reconnect" to form a gap in the vicinity of the original intersecting point, as shown in Fig. . Therefore the original intersecting point, Eq. (242), gives the approximate location of the gap. Furthermore, using Eq. (242), we can determine the frequency of the intersecting point, which is given by

$$\omega^2 = k_{\parallel m}^2 V_A^2 = \left(\frac{n}{2m+1} \right)^2 \frac{V_A^2}{R_0^2}, \tag{243}$$

which can be further written

$$\omega = \frac{1}{2q_{\text{gap}}} \frac{V_A}{R_0}. \tag{244}$$

According to the same reasoning given in the above, Eq. (244) is an approximation to the center frequency of the TAE gap. The frequency and the location given above are also an approximation to the frequency and location of the TAE modes that lie in the gap.

For the ellipticity-induced gap (EAE gap), which is formed due to the coupling of m and $m+2$ harmonics, the location is approximately determined by

$$k_{\parallel m} = -k_{\parallel m+2}, \tag{245}$$

which gives

$$q_{\text{gap}} = \frac{m+1}{n}, \tag{246}$$

and the approximate center angular frequency is

$$\omega^2 = k_\parallel^2 V_A^2 = \left(\frac{n}{m+1}\right)^2 \frac{V_A^2}{R_0^2}, \tag{247}$$

which can be written as

$$\omega = \frac{2}{2q_{\text{gap}}} \frac{V_A}{R_0}.$$

Generally, for the gap formed due to the coupling of m and $m+\Delta$ harmonics, we have

$$k_{\parallel m} = -k_{\parallel m+\Delta},$$

which gives

$$q_{\text{gap}} = \frac{2m+\Delta}{2n}, \tag{248}$$

and

$$\omega^2 = k_\parallel^2 V_A^2 = \left(\frac{\Delta n}{2m+\Delta}\right)^2 \frac{V_A^2}{R_0^2}. \tag{249}$$

Equation (249) can also be written as

$$\omega = \frac{\Delta}{2q_{\text{ gap}}} \frac{V_A}{R_0}. \tag{250}$$

7 Shooting method for finding global eigenmodes

After using Fourier spectrum expansion and taking the inner product over θ, Eq. (190) can be written as the following system of ordinary differential equations:

$$\frac{d}{d\psi}\begin{pmatrix} \overline{P}_1^{(1)}(\psi) \\ \vdots \\ \overline{P}_1^{(L)}(\psi) \\ \xi_\psi^{(1)}(\psi) \\ \vdots \\ \xi_\psi^{(L)}(\psi) \end{pmatrix} = \begin{pmatrix} A_{11}^{(11)} & \ldots & A_{11}^{(1L)} & A_{12}^{(11)} & \ldots & A_{12}^{(1L)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{11}^{(L1)} & \ldots & A_{11}^{(LL)} & A_{12}^{(L1)} & \ldots & A_{12}^{(LL)} \\ A_{21}^{(11)} & \ldots & A_{21}^{(1L)} & A_{22}^{(11)} & \ldots & A_{22}^{(1L)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{21}^{(L1)} & \ldots & A_{21}^{(LL)} & A_{22}^{(L1)} & \ldots & A_{22}^{(LL)} \end{pmatrix} \begin{pmatrix} \overline{P}_1^{(1)}(\psi) \\ \vdots \\ \overline{P}_1^{(L)}(\psi) \\ \xi_\psi^{(1)}(\psi) \\ \vdots \\ \xi_\psi^{(L)}(\psi) \end{pmatrix}, \tag{251}$$

where L is the total number of the poloidal harmonics included in the Fourier expansion, the matrix elements $A_{\alpha\beta}^{(ij)}$ are functions of ψ and ϖ^2. Next, we specify the boundary condition for the system. Note that equations system (251) has $2L$ first-order differential equations, for which we need to specify $2L$ boundary conditions to make the solution unique. The geometry determines that the radial displacement at the magnetic axis must be zero, i.e.,

$$\xi_\psi^{(l)}(\psi = \psi_0) = 0, \text{ For } l = 1, 2, \ldots, L. \tag{252}$$

We consider only the modes that vanish at the plasma boundary, for which we have the following boundary conditions:

$$\xi_\psi^{(l)}(\psi = \psi_{\text{LCFS}}) = 0, \text{ For } l = 1, 2, \ldots, L \tag{253}$$

Now Eqs. (252) and (253) provide $2L$ boundary conditions, half of which are at the boundary $\psi = \psi_0$ and half are at the boundary $\psi = \psi_{\text{LCFS}}$. Therefore equations system (251) along with the boundary conditions Eqs. (252) and (253) constitutes a standard two-points boundary problem[5]. Note, however, that we are solving a eigenvalue problem, for which there is an additional equation for ϖ^2:

$$\frac{d\varpi^2}{d\psi} = 0. \tag{254}$$

This increases the number of equations by one and so we need one additional boundary condition. Note that, by eliminating all $\overline{P}_1^{(l)}$, equations system (251) can be written as a system of second-order differential equations for $\xi_\psi^{(l)}$. Further note that the unknown functions $\xi_\psi^{(l)}$ satisfy homogeneous equations and homogeneous boundary conditions, which indicates that if $\xi_\psi^{(l)}$ with $l = 1, 2, \ldots L$ are solutions, then $c\xi_\psi^{(l)}$ are also solutions to the original equations, where c is a constant. Therefore the value of the derivative of $d\xi_\psi^{(l)} / d\psi$ at the boundary have one degree of freedom. Due to this fact, one of the derivatives $d\xi_\psi^{(1)} / d\psi$, $d\xi_\psi^{(2)} / d\psi$,..., $d\xi_\psi^{(L)} / d\psi$ at ψ_0 can be set to be a nonzero value. For example, setting the value of $d\xi_\psi^{(1)} / d\psi$ at ψ_0 to be 0.5 and making use of $\xi_\psi^{(l)} = 0$ at ψ_0, we obtain

$$A_{21}^{(11)}(\psi_0)\overline{P}_1^{(1)}(\psi_0) + A_{21}^{(12)}(\psi_0)\overline{P}_1^{(2)}(\psi_0) + \ldots + A_{21}^{(1L)}(\psi_0)\overline{P}_1^{(L)}(\psi_0) = 0.5, \tag{255}$$

which can be solved to give

$$\overline{P}_1^{(L)}(\psi_0) = \frac{1}{A_{21}^{(1L)}(\psi_0)}\Big[0.5 - A_{21}^{(11)}(\psi_0)\overline{P}_1^{(1)}(\psi_0) - A_{21}^{(12)}(\psi_0)\overline{P}_1^{(2)}(\psi_0) - ...\Big], \tag{256}$$

which provides the additional boundary condition we need. In the present version of my code, for convenience, I directly set the value of $\overline{P}_1^{(L)}(\psi_0)$ to a small value, instead of using Eq. (256). The following sketch map describes the function $\mathbf{F}(\mathbf{X})$ for which we need to find roots in the shooting process.

$$\mathbf{X} = \begin{pmatrix} \overline{P}_1^{(1)}(\psi_0) \\ \overline{P}_1^{(2)}(\psi_0) \\ \vdots \\ \overline{P}_1^{(L-1)}(\psi_0) \\ \varpi^2 \end{pmatrix} \longrightarrow \mathbf{F}(\mathbf{X}) = \begin{pmatrix} \xi_\psi^{(1)}(\psi_{\mathrm{LCFS}}) \\ \xi_\psi^{(2)}(\psi_{\mathrm{LCFS}}) \\ \vdots \\ \xi_\psi^{(L-1)}(\psi_{\mathrm{LCFS}}) \\ \xi_\psi^{(L)}(\psi_{\mathrm{LCFS}}) \end{pmatrix} \tag{257}$$

8 Benchmark of GTAW code

To benchmark GTAW code, we use it to calculate the continua and gap modes of the Solovev equilibrium and compare the results with those given by NOVA code. The Solovev equilibrium used in the benchmark case is given by

$$\Psi = \frac{B_0}{2R_0^2\kappa_0 q_0}\Big[R^2Z^2 + \frac{\kappa_0^2}{4}(R^2 - R_0^2)^2\Big], \tag{258}$$

$$p_0 = p_0(0) - \frac{B_0(\kappa_0^2+1)}{\mu_0 R_0^2\kappa_0 q_0}\Psi,\ g = g_0, \tag{259}$$

with $B_0 = 1T$, $R_0 = 1m$, $g_0 = 1mT$, $\kappa_0 = 1.5$, $q_0 = 3$, and $p_0(0) = 1.1751 \times 10^4$Pa. The flux surface with the minor radius being $0.3m$ (corresponding to $\Psi = 2.04 \times 10^{-2}Tm^2$) is chosen as the boundary flux surface. Main plasma is taken to be Deuterium and the number density is taken to be uniform with $n_D = 2 \times 10^{19}m^{-3}$. Figure 3 compares the Alfven continua calculated by NOVA and GTAW, which shows good agreement between them.

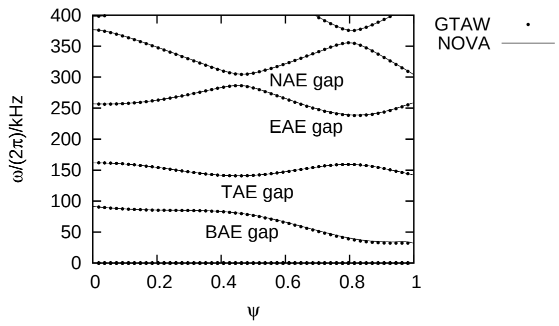

Figure 3. Comparison of the $n = 1$ Alfven continua calculated by NOVA and our code. The continua are calculated in the slow sound approximation[7] and the equilibrium used is the Solovev equilibrium given in Eqs. (258) and (259).

A gap mode with frequency $f = 297$kHz is found in the NAE gap by both NOVA and GTAW. The poloidal mode numbers of the two dominant harmonics are $m = 2$ and $m = 5$, which is consistent with the fact that a NAE is formed due to the coupling between m and $m + 3$ harmonics. Before comparing the radial structure of the poloidal harmonics given by the two codes, a discussion about the assumption adopted in NOVA is desirable. As is pointed out by Dr. Gorelenkov, NOVA at present is restricted to up-down symmetric equilibrium and, for this case, it can be shown that the amplitude of all the radial displacement can be transformed to real numbers. For this reason, NOVA use directly real numbers for the radial displacement in its calculation. In GTAW code, the amplitude of the poloidal harmonics of the radial displacement are complex numbers. The Solovev equilibrium used here is up-down symmetric and the results given by GTAW indicate the poloidal harmonics of the radial displacement can be transformed (by multiplying a constant such as $(1 - i)$) to real numbers. After transforming the radial displacement to real numbers, the results can be compared with those of NOVA. Figure 4 compares the radial structure of the dominant poloidal harmonics ($m = 2, 3, 4, 5$) given by the two codes, which indicates the results given by the two codes agree with each other well.

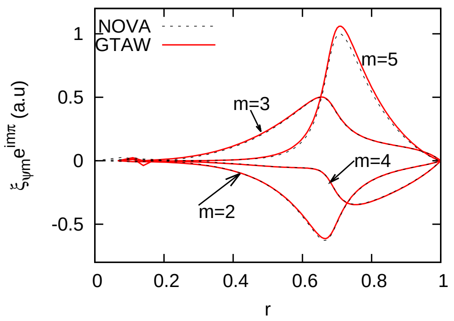

Figure 4. The dominant poloidal harmonics ($m = 2, 3, 4, 5$) of a $n = 1$ NAE as a function of the radial coordinate. The solid lines are the results of GTAW while the dashed lines are those of NOVA. The corresponding poloidal mode numbers are indicated in the figure. The frequency of the mode $f = 297$kHz. The equilibrium is given by Eqs. (258) and (259).

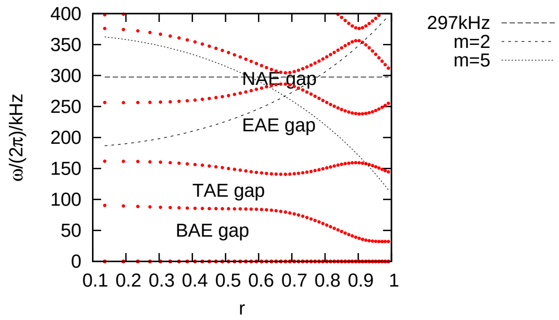

Figure 5. Slow sound approximation of the $n = 1$ continua of the Solovev equilibrium given by Eqs. (258) and (259). Also plotted are the frequency of the NAE ($f = 297$kHz) and the $m = 2$ and $m = 5$ continua in cylindrical limit.

Figure 6 plots the mode structure of the NAE on $\phi = 0$ plane, which shows that the mode has an anti-ballooning structure, i.e., the mode is stronger at the high-field side than at the low-field side.

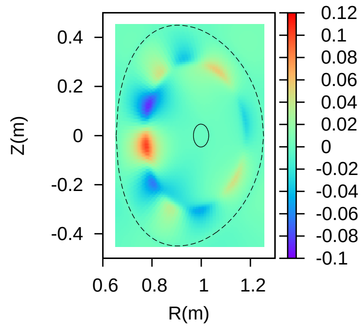

Figure 6. Two dimension mode structure of the NAE in Fig. 4. The dashed line in the figure indicates the boundary magnetic surface and the small circle indicates the inner boundary used in the numerical calculation.

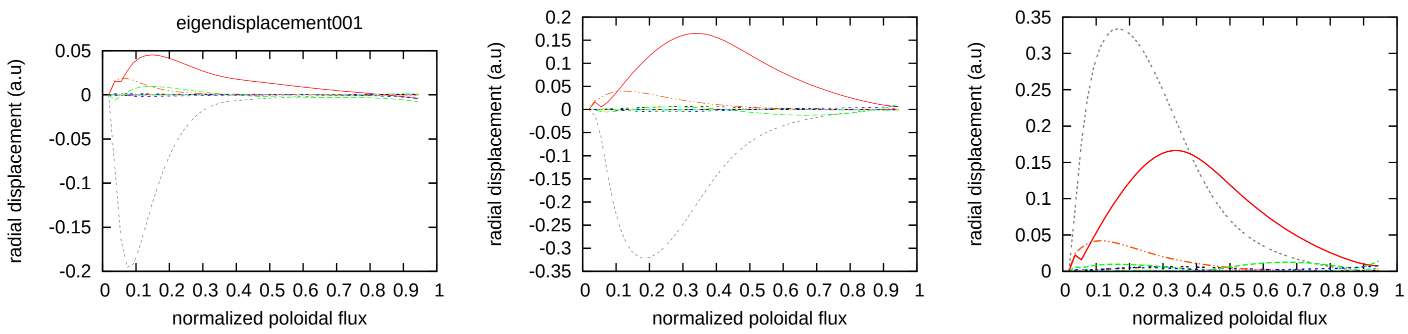

Figure 7. Real part (a), imaginary part (b), and absolute value of the amplitude (c) of the poloidal harmonics of a $n = 1$ TAE as a function of the radial coordinate. The frequency of the mode is $f = 93$kHz. The dominant poloidal harmonics are those with $m = 3$ and $m = 4$. The equilibrium is given by Eqs. (258) and (259).

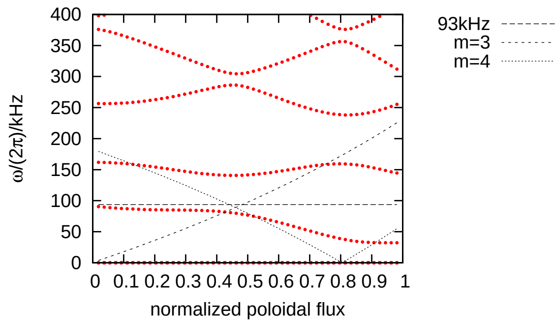

Figure 8. Slow sound approximation of the continua of the Solovev equilibrium. Also plotted are the frequency of the TAE ($f = 93$kHz) and the $m = 3$ and $m = 4$ continua in cylindrical limit. Toroidal mode number $n = 1$.

For the case that $p_0(0) = 1.5 \times 10^4$Pa, a TAE with $f = 102$kHz is found in the TAE gap. The radial dependence of the poloidal harmonics of the mode is plotted in Fig. 9. Figure 10 plots the frequency of the mode on the Alfven continua graphic.

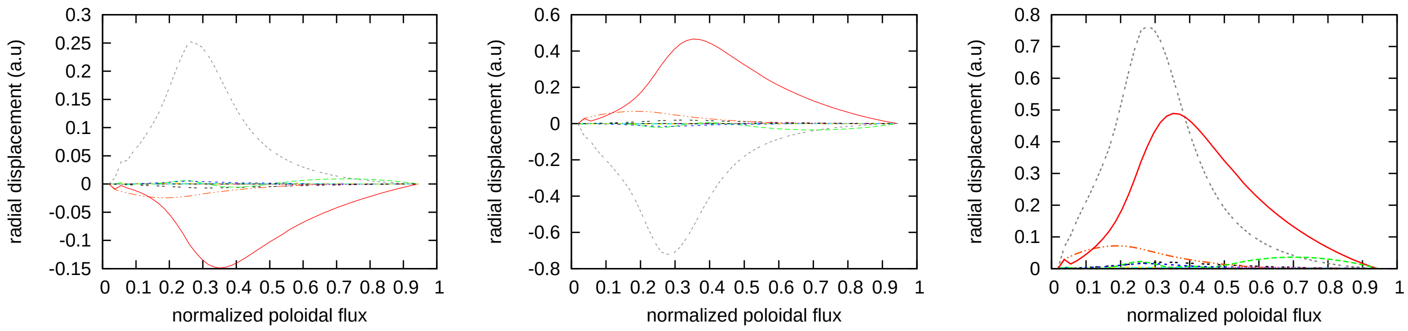

Figure 9. Real part (a), imaginary part (b), and absolute value of the amplitude (c) of the poloidal harmonics of a $n = 1$ TAE as a function of the radial coordinate. The frequency of the mode is $f = 102$kHz. The dominant poloidal harmonics are those with $m = 3$ and $m = 4$. The equilibrium is given by Eqs. (258) and (259), (old)

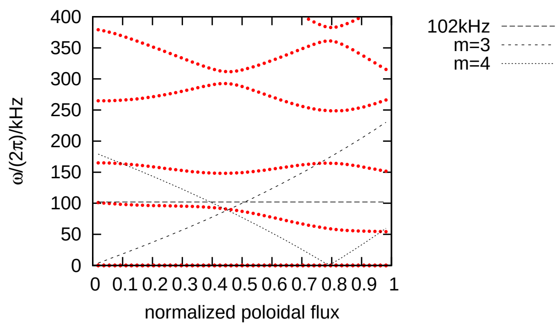

Figure 10. Slow sound approximation of the continua of the Solovev equilibrium. Also plotted are the frequency of the TAE ($f = 102$kHz) and the $m = 3$ and $m = 4$ continua in cylindrical limit. Toroidal mode number $n = 1$.(old)

9 Numerical results for EAST tokamak

The content in this section has been published in my 2014 paper[8].

9.1 EAST Tokamak equilibrium

The tokamak equilibrium used in this paper is reconstructed by EFIT code by using the information of profiles measured in EAST experiment[9]. The shape of flux surfaces within the last-closed-flux surface (LCFS) are plotted in Fig. 11, where $\theta =$ const curves are also plotted. In the paper, I said that the equilibrium was a double-null configuration with the LCFS connected to the lower X point. This is wrong. The configuration with the LCFS connected to the lower X point should be called lower single null configuration. The double-null configuration is a configuration with LCFS connected to both the lower and upper X points. In practice, if the spacial seperation between the flux surface connected to the low X point and the flux surface connected to the upper X point, dRsep, is smaller than a value (e.g. 1cm), the configuration can be considered as a double null configuration, where dRsep is the spacial separation between the two flux surfaces on the low-field side of the midplane.

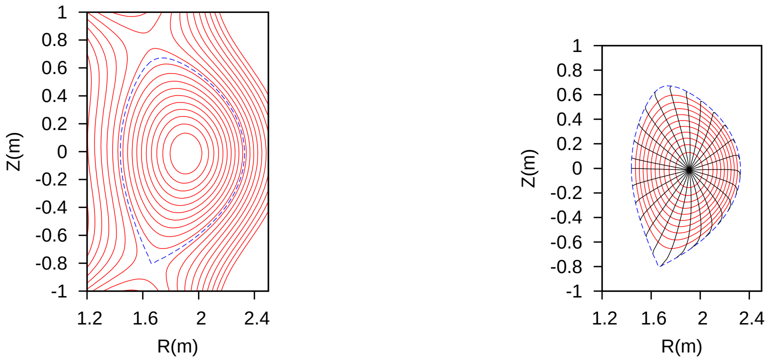

Figure 11. Grid points (the intersecting points of two curves in the figure) corresponding to uniform poloidal flux and uniform poloidal arc length for EAST discharge #38300 at 3.9s (G-file name: g038300.03900, which was provided by Dr. Guoqiang Li).

The profiles of safety factor, pressure, and electron number density are plotted in Fig. 12.

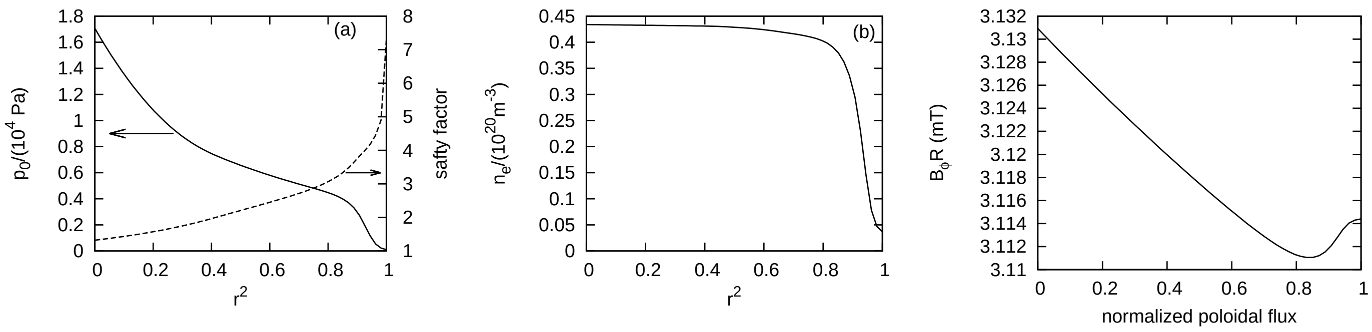

Figure 12. Safety factor and pressure (a), toroidal field function (b), and electron number density (c) as a function of the radial coordinate for EAST discharge #38300 at 3.9s (G-eqdsk filename g038300.03900).

The mass density ρ_0 is calculated from $\rho_0 = m_i n_i$, where m_i is the mass of the main ions (deuterium ions in this discharge), n_i is the number density of the ions, which is inferred from n_e by using the neutral condition $n_i = n_e$ (impurity ions are neglected).

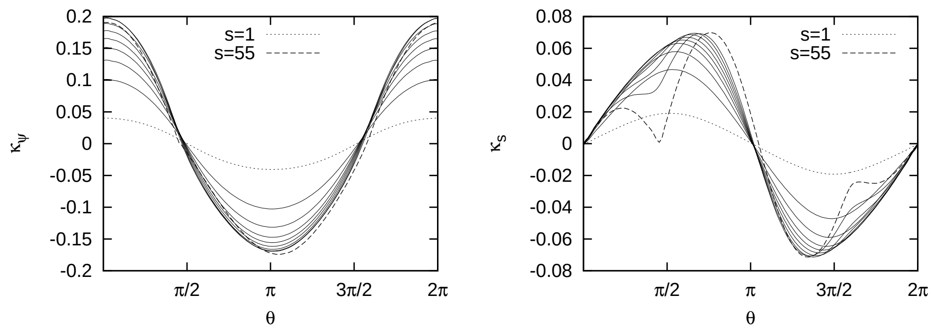

Figure 13. Normal and geodesic magnetic curvature as a function of the poloidal angle. Different lines in the figure correspond to different magnetic surfaces.

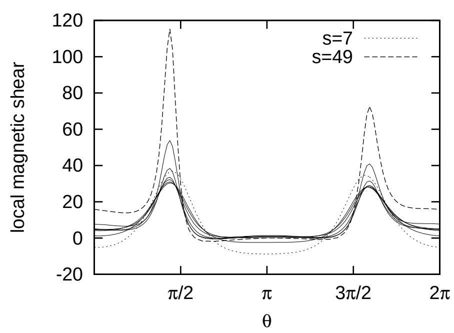

Figure 14. Negative local magnetic shear S as a function of the poloidal angle. The different lines correspond to different magnetic surfaces. The equilibrium is for EAST shot #38300 at 3.9s.

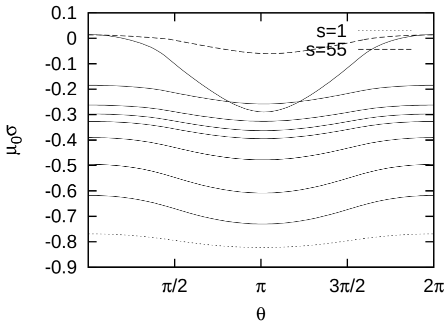

Figure 15. $\mu_0\sigma$ as a function of the poloidal angle. The different lines correspond to the values of $\mu_0\sigma$ on different magnetic surfaces

9.2 Numerical results of MHD continua

The eigenfrequency of Eq. (213), ϖ^2, as a function of the radial coordinate gives the continua for the equilibrium. It can be proved analytically that the eigenfrequency of Eq. (213), ϖ^2, is a real number (I do not prove this). Making use of this fact, we know that a crude method of finding the eigenvalue of Eq. (213) is to find the zero points of the real part of the determinant of $\mathbf{E}$. Since, in this case, both the independent variables and the value of the function are real, the zero points can be found by using a simple one-dimension root finder. This method was adopted in the older version of GTAW (bisection method is used to find roots). In the latest version of GTAW, as mentioned above, the generalized eigenvalue problem in Eq. (213) is solved numerically by using the **zggev** subroutine in Lapack library. (The eigenvalue problem is solved without the assumption that ϖ^2 is real number. The eigenvalue ϖ^2 obtained from the routine is very close to a real number, which is consistent with the analytical conclusion that the eigenvalue ϖ^2 must be a real number.)

Figure 16 plots the eigenfrequency of Eq. (213) as a function of the radial coordinate $\overline{\Psi}$. The result is calculated in the slow sound approximation, thus giving only the Alfven branch of the continua. Also plotted in Fig. 16 are the Alfven continua in the cylindrical limit. As shown in Fig. 16, the Alfven continua in toroidal geometry do not intersect each other, thus forming gaps at the locations where the cylindrical Alfven continua intersect each other.

The first gap, which is formed due to the coupling of sound wave and Alfven wave, starts from zero frequency. This gap is called BAE gap since beta-induced Alfven eigenmode (BAE) can exist in this gap. The second gap is called TAE gap, which is formed mainly due to the coupling of m and $m+1$ poloidal harmonics. The third gap is called EAE gap, which is formed mainly due to the coupling of m and $m+2$ poloidal harmonics. The fourth gap is called NAE gap, which is formed due to the coupling of m and $m+3$ poloidal harmonics. A gap can be further divided into sub-gaps according to the two dominant poloidal harmonics that are involved in forming the gap. For example, a sub-gap of the TAE gap is the one that is formed mainly due to the coupling of $m=1$ and $m=2$ harmonics. For the ease of discussion, we call this sub-gap "(1, 2) sub-gap", where the two numbers stand for the poloidal mode numbers. The frequency range of a sub-gap is defined by the frequency difference of the two extreme points on the continua. The radial range of the sub-gap can be defined as the radial region whose center is the location of one of the extreme points on the continua, width is the half width between the neighbor left and right extreme points.

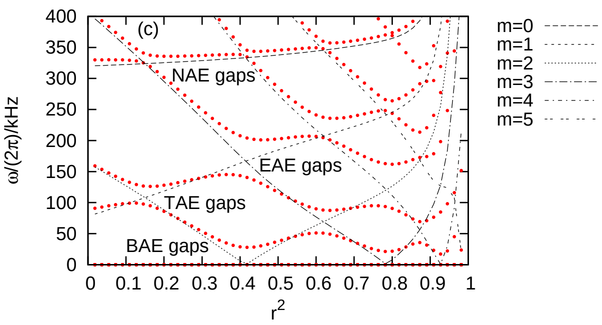

Figure 16. $n=1$ Alfven continua in toroidal geometry (red dots)(calculated in slow sound approximation) and in cylindrical geometry limit for $m=0,1,2,3,4$, and 5 (calculated by using Eq. (220)). The equilibrium used for this calculation is for EAST shot #38300 at 3.9s (G-eqdsk filename g038300.03900, which was provided by Dr. Guoqiang Li). (main ions are deuterium, impurity ions are assumed to be absent).

Figure 17 compares the continua of the full ideal MHD model with those of slow sound and zero β approximations. The results indicate that the slow sound approximation eliminates the sound continua while keeps the Alfven continua almost unchanged. The zero β approximation eliminates the BAE gap.

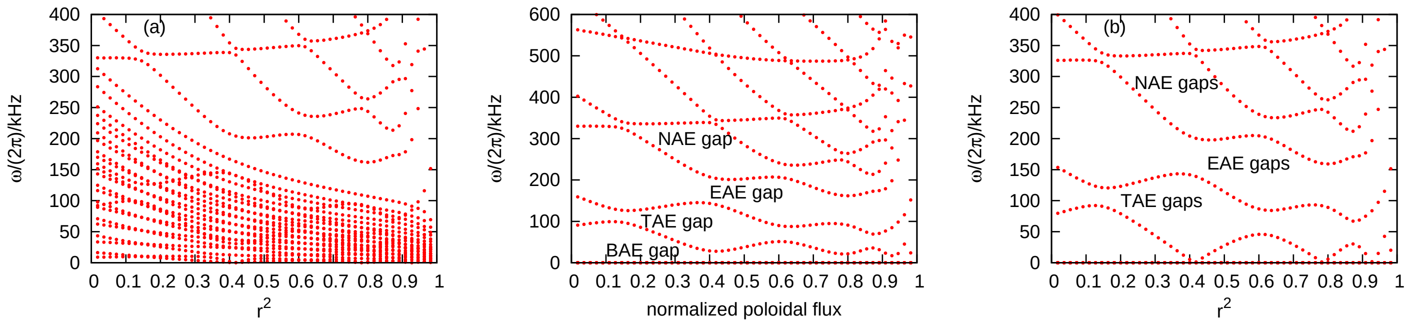

Figure 17. (a) full continua (b) slow sound approximation of the continua (c) zero β approximation of the continua. Other parameters: toroidal mode number $n=1$, the range of poloidal harmonics number is truncated within $[-10{:}\ 10]$. The equilibrium used for this calculation is for EAST shot 38300 at 3.9s.

(Numerical results indicate that the eigenvalue ϖ^2 is always grater than or equal to zero. Can this point be proved analytically?)

In order to verify the numerical convergence about the number of the poloidal harmonics included in the expansion, we compares the results obtained when the poloidal harmonic numbers are truncated in the range $[-10, 10]$ and those obtained when the truncation region is $[-15, 15]$. The results are plotted in Fig. 18, which shows that the two results agree with each other very well for the low order continua in the core region of the plasma. For continua in the edge region or higher order continua, there are some discrepancies between the two results. These discrepancies are due to that higher order poloidal harmonics are needed in evaluating the continua for those cases.

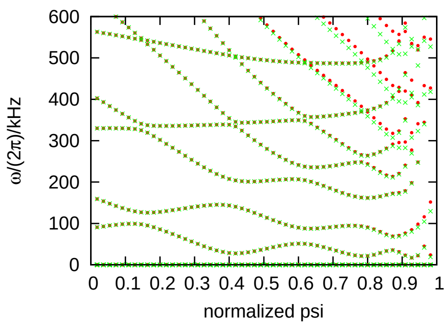

Figure 18. Comparison of the results obtained when the poloidal harmonic numbers are truncated in the range $[-10, 10]$ (solid circles) and those obtained when the truncation region is $[-15, 15]$ (cross marks). The equilibrium used for this calculation is for EAST shot #38300 at 3.9s.

The $n=4$ Alfven continua are plotted in Fig. 19, which shows that there are more TAE gaps than those of the $n=1$ case. The number of gaps is roughly given by $n(q_{\text{edge}} - q_{\text{axis}})$ for a monotonic q profile[10].

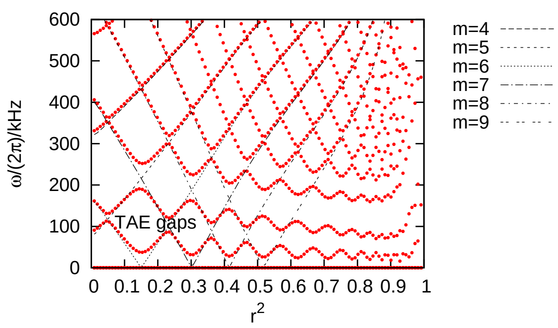

Figure 19. $n=4$ Alfven continua (in slow sound approximation). The poloidal harmonic numbers are truncated in the range $[-15, 20]$. The equilibrium used for this calculation is for EAST shot 38300 at 3.9s. The corresponding Alfven continua in the cylindrical limit are also plotted.

Remarks: If, instead of the definition (169), we define the $\langle \ldots \rangle_{m'm}$ operator as

$$\langle W \rangle_{m'm} = \frac{1}{2\pi} \int_0^{2\pi} W e^{i(m+m')\theta} d\theta, \tag{260}$$

then, can we still obtain correct results for the continuous spectrum? When I began to work on the calculation of the continuous spectrum, I noticed that the definition (169), instead of (260), is adopted in Cheng's paper[3]. By intuition, I thought the definition (260) should work as well as (169). Since the definition (260) is simpler than (169) (there is no additional minus in (260)), I prefer using the definition (260). This choice does not cause me trouble when I use symmetrical truncation of the poloidal harmonics. Trouble appears when I try using asymmetrical truncation of the poloidal harmonics. For example, when asymmetrical truncation of the poloidal harmonics is used (e.g. poloidal harmonics in the range $[-10, 15]$), it is easy to verify analytically that the determinant of the resulting matrix for the case $\gamma = 0$ is zero for any values of ω^2. It is obvious that this will not give correct results for the continuous spectrum. It took me two days to find out the definition in (169) must be used in order to deal with asymmetrical truncation of poloidal harmonics. In summary, the advantage of using the definition (169) over (260), is that the former can deal with the asymmetrical truncation of the poloidal harmonics, while the latter is limited to the case of symmetrical truncation.

9.3 Numerical results of global modes

When analyzing the modes calculated numerically, we need to distinguish two kinds of modes: the continuum modes and the gap modes. In principle, the continuum mode is defined as the mode whose frequency is within the Alfven continua while the gap mode is defined as the mode whose frequency is within the frequency gap of the Alfven continua. However, for realistic equilibria, any given frequency will touch the continua at one of the radial locations.

However, for realistic structure of continua, both the range and the center of the frequency of a gap change with the radial coordinate, as is shown in Fig. 16. As a result, a given frequency usually can not be within a gap for all radial locations, i.e., the frequency usually intersects the Alfven continua at some radial locations. These locations are the Alfven resonant surfaces. As is pointed out in Ref. [?], the mode structure have singularity given by $[c_1 \ln|\psi - \psi_s| + c_2]$ at the resonant surface, where ψ_s is the radial coordinate of the resonant surface and c_2 can have finite discontinuity.

In practice, continuum modes can be easily distinguished from gap modes by examining the radial structure of the poloidal harmonics of the mode. If the radial mode structure has dominant singularities at the Alfven resonant surfaces, then the modes are continua modes. If the dominant peaks of the mode are not at the Alfven resonant surfaces, then the mode is probably a gap mode (further confirmation can be obtained by examining poloidal mode number of the dominant harmonics, discussed later). The mode structure of gap modes can also have singular peak at the resonant surface, but the peak is usually smaller than the dominant peak.

Mode structure of a $n=1$ TAE mode is plotted in Fig. 20

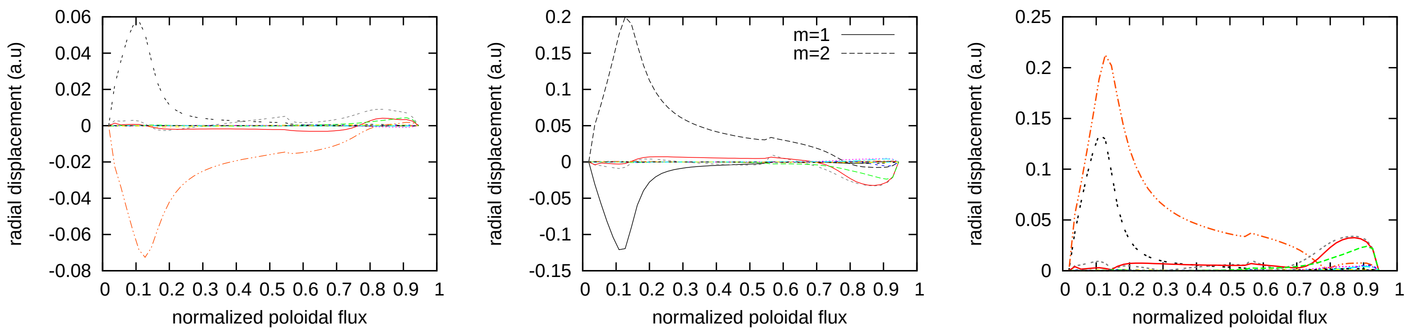

Figure 20. Real part (a), imaginary part (b), and amplitude (c) of the poloidal harmonics of a $n = 1$ TAE mode as a function of the radial coordinate. The frequency of the mode is $f = 101$kHz. The poloidal harmonics with $m = 1$ and $m = 2$ are dominant. The equilibrium used for this calculation is for EAST shot 38300 at 3.9s (G-file name: g038300.03900). (D)

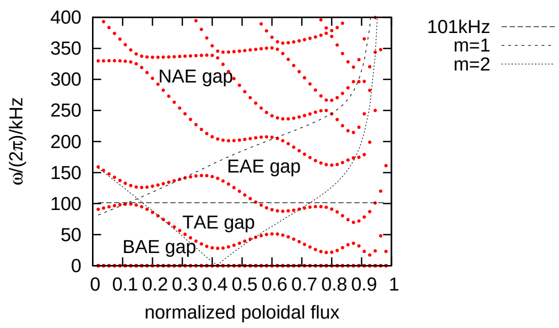

Figure 21. Slow sound approximation of the continua. Also plotted are the frequency of the TAE mode ($f = 101$kHz) and the $m = 1$ and $m = 2$ continua in cylindrical limit. Toroidal mode number $n = 1$. The equilibrium is the same as Fig (D)

Fig. 22. plots the radial mode structure of another TAE with frequency $f = 61.91\text{kHz}$.

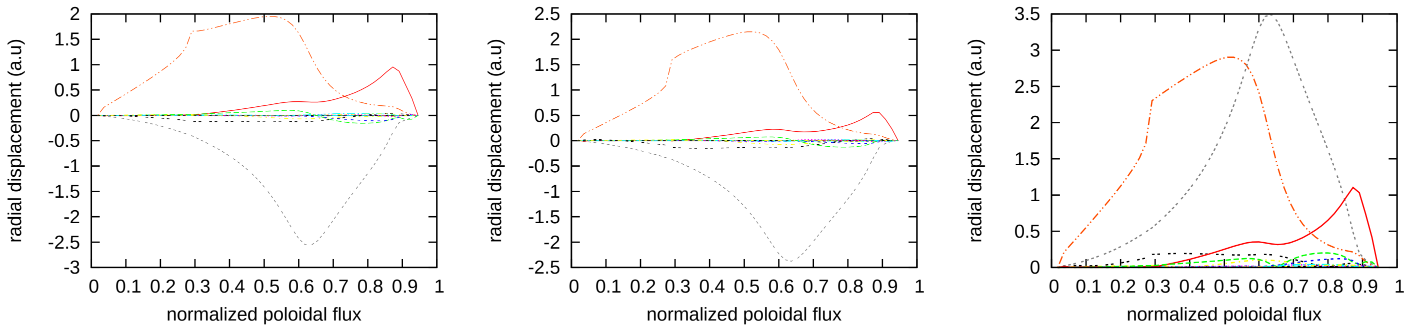

Figure 22. Real part (a), imaginary part (b), and amplitude (c) of the poloidal harmonics of a $n=1$ TAE mode as a function of the radial coordinate. The frequency of the mode is $f = 61.91\text{kHz}$. The poloidal harmonics with $m=2$ and $m=3$ are dominant. The equilibrium used for this calculation is for EAST shot 38300 at 3.9s (G-file name: g038300.03900). (D)

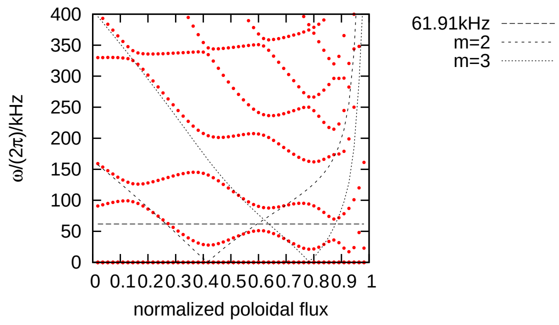

Figure 23. Slow sound approximation of the continua. Also plotted are the frequency of the TAE mode ($f = 61.91\text{kHz}$) and the $m=2$ and $m=3$ continua in cylindrical limit. Toroidal mode number $n=1$. The equilibrium is the same as Fig (D)

An example of ellipticity-induced Alfven Eigenmode (EAE) is plotted in Fig. 24. The mode is identified as an EAE mode because it satisfies the following three requirements: (1) the mode has two dominant harmonics with poloidal mode number differing by two ($m=2$ and $m=4$ for this case); (2) the frequency of the mode $f = 130\text{kHz}$ is within in the continuum gap formed due to the coupling of these two poloidal harmonics; (3) the location of the peak of the radial mode structure is within the continuum gap.

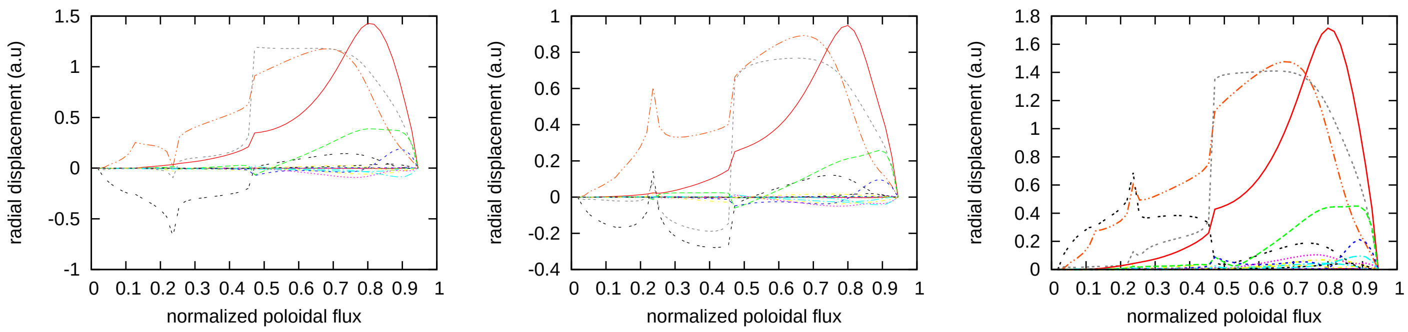

Figure 24. Real part (a), imaginary part (b), and amplitude (c) of the poloidal harmonics of a $n=1$ EAE mode as a function of the radial coordinate. The poloidal harmonics with $m=2$ and $m=4$ are dominant. $f = 130\text{kHz}$. The equilibrium is for EAST shot 38300 at 3.9s (G-file name: g038300.03900). (D)

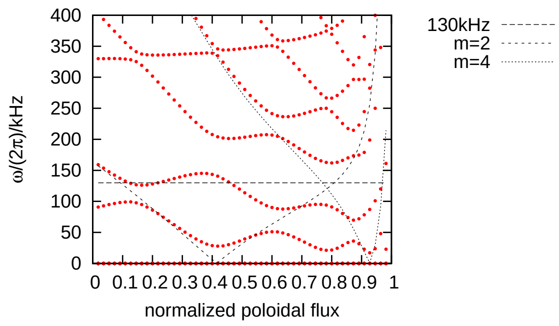

Figure 25. $n=1$ Alfvén continua (D)

An example of non-circularity-induced Alfven Eigenmode (NAE) is plotted in Fig. 26.

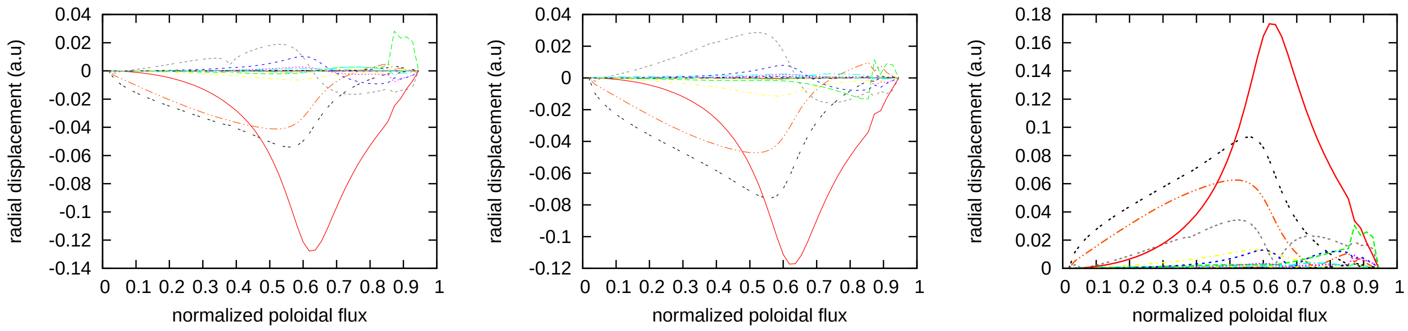

Figure 26. Real part (a), imaginary part (b), and amplitude (c) of the poloidal harmonics of a $n=1$ NAE mode as a function of the radial coordinate. The poloidal harmonics with $m=1$ and $m=4$ are dominant. The frequency of the mode is $f=216$kHz. The equilibrium is for EAST shot 38300 at 3.9s (G-file name: g038300.03900). (D)

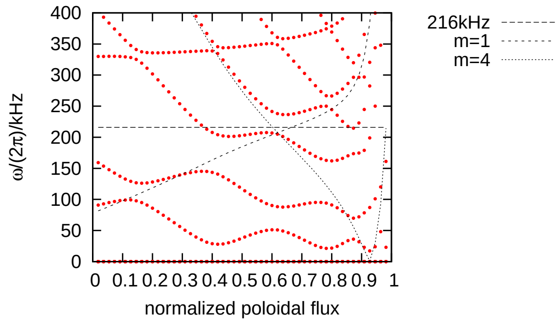

Figure 27. $n=1$ Alfvén continua (D)

$n=4$

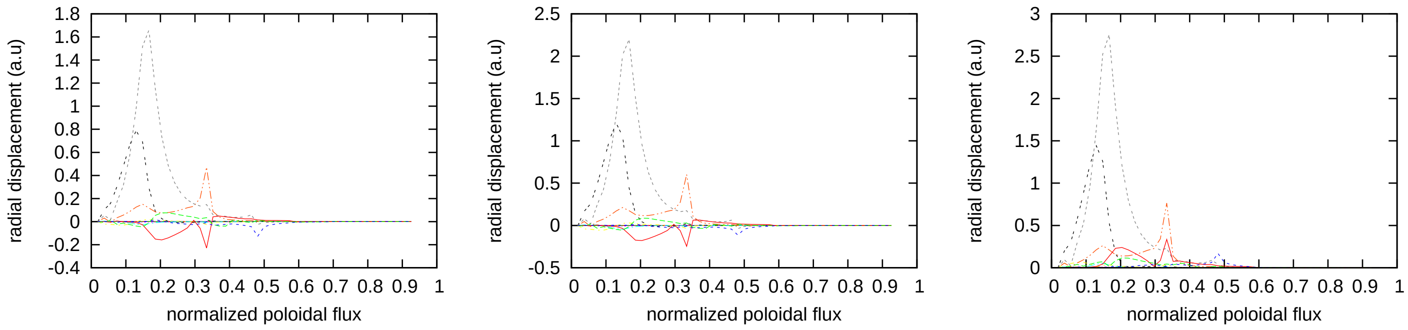

Figure 28. Real part (a), imaginary part (b), and amplitude (c) of the poloidal harmonics of a $n=4$ EAE mode as a function of the radial coordinate. The frequency of the mode is $f=330$kHz. The poloidal harmonics with $m=5$ and $m=7$ are dominant. The poloidal harmonics are truncated into the range $m=[-10,10]$ in the numerical calculation. The equilibrium is for EAST shot 38300 at 3.9s (G-file name: g038300.03900). (H)

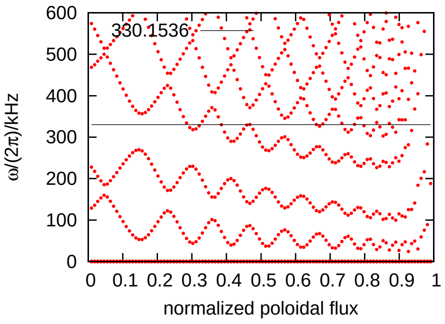

Figure 29. The frequency of the EAE mode given in Fig. 28. (H)

9.4 Magnetic curvature

In this section, we derive formulas for calculating the geodesic curvature and normal curvature in magnetic surface coordinate system (ψ, θ, ϕ). The derivation looks tedious but the final results are compact (especially for the geodesic curvature κ_s). The magnetic curvature is defined by $\boldsymbol{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b}$, which can be further written as

$$\begin{aligned}\boldsymbol{\kappa} &= \frac{1}{B_0} \mathbf{B}_0 \cdot \nabla \frac{\mathbf{B}_0}{B_0} \\ &= \frac{1}{B_0} (\nabla \Psi \times \nabla \phi + g \nabla \phi) \cdot \nabla \frac{\mathbf{B}_0}{B_0} \end{aligned} \tag{261}$$

In magnetic surface coordinate system (ψ, θ, ϕ), equation (261) is written as

$$\begin{aligned}\boldsymbol{\kappa} &= \frac{1}{B_0} (\Psi' \nabla \psi \times \nabla \phi + g \nabla \phi) \cdot \nabla \frac{\mathbf{B}_0}{B_0} \\ &= \frac{\Psi'}{B_0} (\nabla \psi \times \nabla \phi) \cdot \nabla \frac{\mathbf{B}_0}{B_0} + \frac{g}{B_0} \nabla \phi \cdot \nabla \frac{\mathbf{B}_0}{B_0} \\ &= -\frac{\Psi'}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{\mathbf{B}_0}{B_0} \right) + \frac{g}{B_0} \frac{1}{R^2} \frac{\partial}{\partial \phi} \left(\frac{\mathbf{B}_0}{B_0} \right) \\ &= -\frac{\Psi'}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{\Psi' \nabla \psi \times \nabla \phi + g \nabla \phi}{B_0} \right) + \frac{g}{B_0} \frac{1}{R^2} \frac{\partial}{\partial \phi} \left(\frac{\Psi' \nabla \psi \times \nabla \phi + g \nabla \phi}{B_0} \right) \\ &= -\frac{\Psi'^2}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{\nabla \psi \times \nabla \phi}{B_0} \right) - \frac{g \Psi'}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{\nabla \phi}{B_0} \right) + \frac{g \Psi'}{B_0^2 R^2} \frac{\partial}{\partial \phi} (\nabla \psi \times \nabla \phi) - \frac{g^2}{B_0^2} \frac{1}{R^3} \hat{\mathbf{R}} \end{aligned} \tag{262}$$

Using $\nabla \psi = -\frac{R}{\mathcal{J}} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}})$, we obtain $\nabla \psi \times \nabla \phi = -\frac{1}{\mathcal{J}} (Z_\theta \hat{\mathbf{Z}} + R_\theta \hat{\mathbf{R}})$. Using this, the three partial derivatives in the above equation are written respectively as

$$\begin{aligned}\frac{\partial}{\partial \theta} \left(\frac{\nabla \psi \times \nabla \phi}{B_0} \right) &= -(Z_\theta \hat{\mathbf{Z}} + R_\theta \hat{\mathbf{R}}) \frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) - \frac{\mathcal{J}^{-1}}{B_0} (Z_{\theta\theta} \hat{\mathbf{Z}} + R_{\theta\theta} \hat{\mathbf{R}}) \\ &= -\left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta} \right] \hat{\mathbf{R}} - \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta} \right] \hat{\mathbf{Z}}, \end{aligned} \tag{263}$$

$$\frac{\partial}{\partial \theta} \left(\frac{\nabla \phi}{B_0} \right) = \frac{\partial}{\partial \theta} \left(\frac{1}{R B_0} \right) \hat{\boldsymbol{\phi}}, \tag{264}$$

$$\frac{\partial}{\partial \phi} (\nabla \psi \times \nabla \phi) = -\frac{1}{\mathcal{J}} R_\theta \hat{\boldsymbol{\phi}}. \tag{265}$$

Using these, $\boldsymbol{\kappa}$ is written as

$$\begin{aligned}\boldsymbol{\kappa} = & \frac{\Psi'^2}{B_0} \mathcal{J}^{-1} \left\{ \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta} \right] \hat{\mathbf{R}} + \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta} \right] \hat{\mathbf{Z}} \right\} \\ & - \frac{g \Psi'}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{1}{R B_0} \right) \hat{\boldsymbol{\phi}} - \frac{g \Psi'}{B_0^2 R^2} \frac{1}{\mathcal{J}} R_\theta \hat{\boldsymbol{\phi}} - \frac{g^2}{B_0^2} \frac{1}{R^3} \hat{\mathbf{R}} \end{aligned}$$

9.4.1 Expression of normal curvature κ_ψ

The component κ_ψ is defined by $\kappa_\psi = \boldsymbol{\kappa} \cdot \nabla \Psi$. Using Eq. (262), κ_ψ is written as

$$\begin{aligned}\kappa_\psi &= -\frac{\Psi'^3}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta} \left(\frac{\nabla \psi \times \nabla \phi}{B_0} \right) \cdot \nabla \psi + 0 + 0 - \frac{\Psi' g^2}{B_0^2} \frac{1}{R^3} \hat{\mathbf{R}} \cdot \nabla \psi \\ &= -\frac{\Psi'^3}{B_0} \mathcal{J}^{-1} \left\{ \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta} \right] \hat{\mathbf{R}} + \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta} \right] \hat{\mathbf{Z}} \right\} \cdot \left[\frac{R}{\mathcal{J}} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) \right] - \frac{\Psi' g^2}{B_0^2} \frac{1}{R^3} \hat{\mathbf{R}} \cdot \\ & \quad \left[-\frac{R}{\mathcal{J}} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) \right] \\ &= -\frac{\Psi'^3}{B_0} \mathcal{J}^{-1} \frac{R}{\mathcal{J}} \left\{ \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta} \right] Z_\theta - \left[\frac{\partial}{\partial \theta} \left(\frac{\mathcal{J}^{-1}}{B_0} \right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta} \right] R_\theta \right\} + \frac{\Psi' g^2}{B_0^2} \frac{1}{R^2} \frac{1}{\mathcal{J}} Z_\theta \\ &= -\frac{\Psi'^3}{B_0} \mathcal{J}^{-1} \frac{R}{\mathcal{J}} \frac{\mathcal{J}^{-1}}{B_0} (R_{\theta\theta} Z_\theta - R_\theta Z_{\theta\theta}) + \frac{\Psi' g^2}{B_0^2} \frac{1}{R^2} \frac{1}{\mathcal{J}} Z_\theta \\ &= -\Psi'^3 R \frac{\mathcal{J}^{-3}}{B_0^2} (R_{\theta\theta} Z_\theta - R_\theta Z_{\theta\theta}) + \frac{\Psi' g^2}{B_0^2} \frac{1}{R^2} \frac{1}{\mathcal{J}} Z_\theta \qquad (266) \\ &= -\Psi'^3 R \frac{\mathcal{J}^{-3}}{B_0^2} Z_\theta^2 \frac{\partial}{\partial \theta} \left(\frac{R_\theta}{Z_\theta} \right) + \frac{\Psi' g^2}{B_0^2} \frac{1}{R^2} \frac{1}{\mathcal{J}} Z_\theta. \qquad (267) \end{aligned}$$

Equation (266) is used in GTAW code to calculate κ_ψ. Equation (267) is not suitable for numerical calculation because Z_θ, which appears both in numerator and denominator, can be very small, leading to significant errors in the numerical results. [My notes: the bad results calculated by Eq. (267) in my code reminded me that Eq. (266) may be better. I switch back to adopt Eq. (266) and the results clearly show that the results given by Eq. (266) are indeed better than those of Eq. (267), as shown in Fig. 30.]

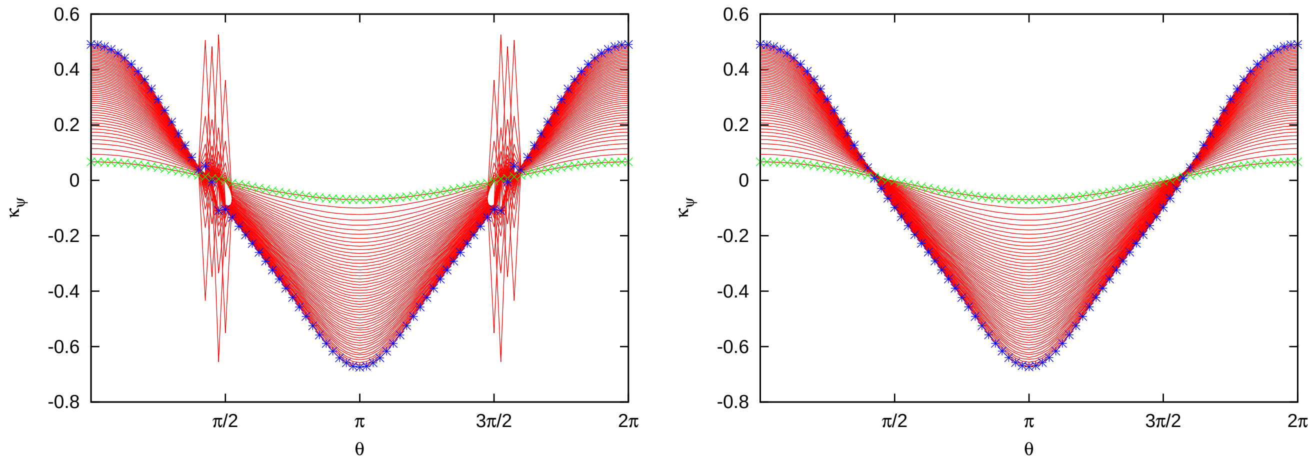

Figure 30. The normal curvature κ_ψ calculated by Eq. (267) (left) and Eq. (266) (right) as a function of the poloidal angle. The different lines corresponds to different magnetic surfaces. The stars correspond to the values of κ_ψ on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is a Solovev equilibrium.

9.4.2 Expression of geodesic curvature κ_s

Next, consider the calculation of the surface component of $\boldsymbol{\kappa}$, the geodesic curvature κ_s, which is defined by

$$\kappa_s \equiv \boldsymbol{\kappa} \cdot \frac{\mathbf{B}_0 \times \nabla \Psi}{B_0^2}. \tag{268}$$

Using

$$\begin{aligned} \mathbf{B}_0 \times \nabla \Psi &= [\nabla \Psi \times \nabla \phi + g \nabla \phi] \times \nabla \Psi \\ &= \Psi'^2 |\nabla \psi|^2 \nabla \phi - g \Psi' \nabla \psi \times \nabla \phi \\ &= \Psi'^2 |\nabla \psi|^2 \frac{1}{R} \hat{\boldsymbol{\phi}} + \frac{g \Psi'}{\mathcal{J}} Z_\theta \hat{\mathbf{Z}} + \frac{g \Psi'}{\mathcal{J}} R_\theta \hat{\mathbf{R}} \end{aligned} \tag{269}$$

and Eq. (263), we obtain

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(\frac{\nabla \psi \times \nabla \phi}{B_0}\right) \cdot \frac{\mathbf{B}_0 \times \nabla \Psi}{B_0^2} &= \left\{ -\left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta}\right] \hat{\mathbf{R}} - \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta}\right] \hat{\mathbf{Z}} \right\} \cdot \frac{g \Psi'}{B_0^2 \mathcal{J}} (Z_\theta \hat{\mathbf{Z}} + R_\theta \hat{\mathbf{R}}) \\ &= -\frac{g \Psi'}{B_0^2 \mathcal{J}} \left\{ \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta}\right] Z_\theta + \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta}\right] R_\theta \right\} \end{aligned} \tag{270}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\nabla \phi}{B_0}\right) \cdot \frac{\mathbf{B}_0 \times \nabla \Psi}{B_0^2} = \frac{\partial}{\partial \theta}\left(\frac{1}{R B_0}\right) \frac{\Psi'^2 |\nabla \psi|^2}{B_0^2 R}. \tag{271}$$

$$\frac{\partial}{\partial \phi}(\nabla \psi \times \nabla \phi) \cdot \frac{\mathbf{B}_0 \times \nabla \Psi}{B_0^2} = -\frac{1}{\mathcal{J}} R_\theta \frac{\Psi'^2 |\nabla \psi|^2}{B_0^2 R}. \tag{272}$$

Using Eqs. (270), (271), and (272), equation (268) is written as

$$\begin{aligned} \kappa_s = & -\frac{\Psi'^2}{B_0} \mathcal{J}^{-1} \left(-\frac{g \Psi'}{B_0^2 \mathcal{J}}\right) \left\{ \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) Z_\theta + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta}\right] Z_\theta + \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) R_\theta + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta}\right] R_\theta \right\} \\ & -\frac{g \Psi'}{B_0} \mathcal{J}^{-1} \frac{\partial}{\partial \theta}\left(\frac{1}{R B_0}\right) \frac{\Psi'^2 |\nabla \psi|^2}{B_0^2 R} - \frac{g \Psi'}{B_0^2} \frac{1}{R^2} \frac{1}{\mathcal{J}} R_\theta \frac{\Psi'^2 |\nabla \psi|^2}{B_0^2 R} - \frac{g^2}{B_0^2} \frac{1}{R^3} \frac{g \Psi'}{B_0^2 \mathcal{J}} R_\theta \end{aligned} \tag{273}$$

The terms in the first line of Eq. (273) is written as

$$\begin{aligned} & \frac{g \Psi'^3}{B_0^3} \mathcal{J}^{-2} \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) Z_\theta^2 + \frac{\mathcal{J}^{-1}}{B_0} Z_{\theta\theta} Z_\theta + \frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) R_\theta^2 + \frac{\mathcal{J}^{-1}}{B_0} R_{\theta\theta} R_\theta\right] \\ = & \frac{g \Psi'^3}{B_0^3} \mathcal{J}^{-2} \left[\frac{\partial}{\partial \theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) (Z_\theta^2 + R_\theta^2) + \frac{\mathcal{J}^{-1}}{B_0} (Z_{\theta\theta} Z_\theta + R_{\theta\theta} R_\theta)\right] \end{aligned} \tag{274}$$

Noting that

$$|\nabla \psi|^2 = \frac{R^2}{\mathcal{J}^2} (Z_\theta^2 + R_\theta^2), \tag{275}$$

and

$$\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right) = 2Z_\theta Z_{\theta\theta} + 2R_\theta R_{\theta\theta} \tag{276}$$

expression (274) is written as

$$\begin{aligned}
&\frac{g\Psi'^3}{B_0^3}\mathcal{J}^{-2}\left[\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right)\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2} + \frac{\mathcal{J}^{-1}}{B_0}\frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right)\right] \\
=&\frac{g\Psi'^3}{B_0^3}\frac{|\nabla\psi|^2}{R^2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) + \frac{g\Psi'^3}{B_0^4}\mathcal{J}^{-3}\frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right)
\end{aligned} \tag{277}$$

The first two terms on the second line of Eq. (273) can be written as

$$-\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\mathcal{J}^{-1}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R} + \frac{1}{B_0R^3}R_\theta\right] \tag{278}$$

The sum of the expression (278) and the first term of expression (277) is written as

$$\begin{aligned}
&\frac{g\Psi'^3}{B_0^3}\frac{|\nabla\psi|^2}{R^2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right) - \frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\mathcal{J}^{-1}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R} + \frac{1}{B_0R^3}R_\theta\right] \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left[\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0}\right)\frac{1}{R^2} - \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R} - \frac{\mathcal{J}^{-1}}{B_0R^3}R_\theta\right] \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left[\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0R^2}\right) - \frac{\mathcal{J}^{-1}}{B_0}\frac{\partial}{\partial\theta}\left(\frac{1}{R^2}\right) - \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R} - \frac{\mathcal{J}^{-1}}{B_0R^3}R_\theta\right] \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left[\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0R^2}\right) + \frac{2\mathcal{J}^{-1}}{B_0R^3}R_\theta - \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R} - \frac{\mathcal{J}^{-1}}{B_0R^3}R_\theta\right] \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left[\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0R^2}\right) + \frac{\mathcal{J}^{-1}}{B_0R^3}R_\theta - \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R}\right] \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left\{\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0R^2}\right) - \mathcal{J}^{-1}\left[-\frac{1}{B_0R^3}R_\theta + \frac{\partial}{\partial\theta}\left(\frac{1}{RB_0}\right)\frac{1}{R}\right]\right\} \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left\{\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^{-1}}{B_0R^2}\right) - \mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{1}{R^2B_0}\right)\right\} \\
=&\frac{g\Psi'^3}{B_0^3}|\nabla\psi|^2\left\{\frac{1}{B_0R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1})\right\}
\end{aligned} \tag{279}$$

Using the above results, κ_s is written as

$$\begin{aligned}
\kappa_s &= \frac{g\Psi'^3}{B_0^4}\mathcal{J}^{-3}\frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right) - \frac{\Psi' g^3}{R^3B_0^4}\mathcal{J}^{-1}R_\theta + \frac{g\Psi'^3}{B_0^4R^2}|\nabla\psi|^2\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) \\
&= \frac{g\Psi'^3}{B_0^4}\frac{1}{\mathcal{J}^2}\left[\frac{1}{2}\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right) + \frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1})\right] - \frac{\Psi' g^3}{R^3B_0^4}\mathcal{J}^{-1}R_\theta \qquad (280)\\
&= \frac{g\Psi'^3}{B_0^4}\frac{1}{\mathcal{J}^2}\left[\frac{1}{2}\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\right) + \frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1})\right] - \frac{\Psi' g^3}{R^3B_0^4}\mathcal{J}^{-1}R_\theta \\
&= \frac{g\Psi'}{B_0^4}\frac{1}{\mathcal{J}^2}\left[\frac{1}{2}\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\right) + \frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1})\right] - \frac{\Psi' g^3}{R^3B_0^4}\mathcal{J}^{-1}R_\theta \\
&= \frac{g\mathcal{J}^{-1}\Psi'}{B_0^3}\left[\frac{1}{B_0\mathcal{J}}\frac{1}{2}\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\right) + \frac{1}{B_0\mathcal{J}}\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) - \frac{g^2}{R^3B_0}R_\theta\right] \qquad (281)
\end{aligned}$$

Using

$$\begin{aligned}
\frac{\partial B_0}{\partial\theta} &= \frac{\partial}{\partial\theta}\sqrt{\left(\frac{|\nabla\Psi|}{R}\right)^2 + \left(\frac{g}{R}\right)^2} \\
&= \frac{\nabla\Psi}{B_0R}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|}{R}\right) - \frac{g^2}{R^3B_0}R_\theta,
\end{aligned}$$

equation (281) is written

$$\kappa_s = \frac{g\mathcal{J}^{-1}\Psi'}{B_0^3}\left[\frac{1}{B_0\mathcal{J}}\frac{1}{2}\mathcal{J}^{-1}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\right) + \frac{1}{B_0\mathcal{J}}\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) + \frac{\partial B_0}{\partial\theta} - \frac{\nabla\Psi}{B_0R}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|}{R}\right)\right]$$

Excluding the $\partial B_0/\partial\theta$ term, the other terms on the r.h.s of the above equation are written

$$\begin{aligned}
&\frac{1}{2}\mathcal{J}^{-2}\frac{\partial}{\partial\theta}\left(\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\right) + \frac{\mathcal{J}|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) - \frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2}{R^2}\right) \\
=&-\frac{1}{2}\frac{\mathcal{J}^2|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-2}) + \frac{\mathcal{J}|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) \\
=&\frac{|\nabla\Psi|^2}{\mathcal{J}R^2}\frac{\partial}{\partial\theta}(\mathcal{J}) + \frac{\mathcal{J}|\nabla\Psi|^2}{R^2}\frac{\partial}{\partial\theta}(\mathcal{J}^{-1}) \\
=&0
\end{aligned}$$

Therefore equation (281) is written

$$\kappa_s = \frac{g\mathcal{J}^{-1}\Psi'}{B_0^3}\left(\frac{\partial B_0}{\partial \theta}\right), \tag{282}$$

which agrees with the formula given in Ref. [6]. Equation (282) takes a very simple form, and provides a clear physical meaning for the geodesic curvature: κ_s is proportional to the poloidal derivative of the magnetic field strength. Equation (284) indicates that the geodesic curvature is zero for an equilibrium configuration that is uniform in poloidal direction. Note that this formula for κ_s is valid for arbitrary Jacobian. (Remarks: When I derived the formula of κ_s for the first time, I found that κ_s can be written in the simple form given by Eq. (282) for the equa-arc length Jacobian. Later I found that κ_s can also be written in the simple form given by Eq. (282) for the Boozer Jacobian. This makes me realize that the simple form given by Eq. (282) may be universally valid for arbitrary Jacobian. However, I did not verify this then. About two years later, I reviewed this notes and succeeded in providing the derivation given above. The derivation given above seems to be tedious and may be greatly simplified in some aspects. But, at present, the above derivation is the only one that I can provide.)

If we choose the equal-arc Jacobian, then Eq. (280) becomes relatively simple:

$$\kappa_s = \frac{g\Psi'^3}{B_0^4}\frac{1}{\mathcal{J}^2}\left[\left(\frac{\oint d\ell}{2\pi}\right)^2\frac{\partial}{\partial\theta}(\mathcal{J}^{-1})\right] - \frac{\Psi' g^3}{R^3 B_0^4}\mathcal{J}^{-1}R_\theta. \tag{283}$$

This form is implemented in GTAW code. Figure 31 gives the results for κ_s calculated by using Eq. (283). I have verified numerically that the results given by Eqs. (282) and (283) agree with each other.

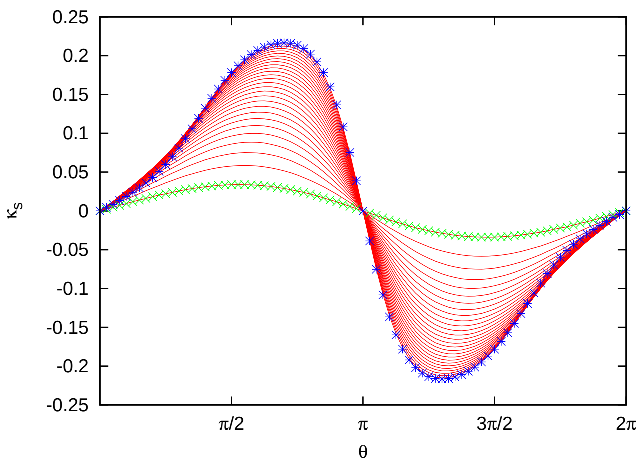

Figure 31. The geodesic magnetic curvature κ_s (calculated by Eq. (283)) as a function of the poloidal angle. The different lines corresponds to different magnetic surfaces. The stars correspond to the values of κ_s on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is a Solovev equilibrium.

9.5 Local magnetic shear

The negative local magnetic shear is defined by

$$S = \left(\nabla\times\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2}\right)\cdot\frac{(\mathbf{B}_0\times\nabla\Psi)}{|\nabla\Psi|^2}. \tag{284}$$

There are two ways of calculating S. The first way is to calculate S in cylindrical coordinate system; the second one is in flux coordinate system. Next, consider the first way. We have

$$\begin{aligned}\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2} &= \frac{(\nabla\Psi\times\nabla\phi + g\nabla\phi)\times\nabla\Psi}{|\nabla\Psi|^2}\\ &= \frac{|\nabla\Psi|^2\nabla\phi + g\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\\ &= \nabla\phi + g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}.\end{aligned} \tag{285}$$

Using this, Eq. (284) is written as

$$\begin{aligned}S &= \left[\nabla\times\left(\nabla\phi + g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\right)\right]\cdot\left[\nabla\phi + g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\right]\\ &= \left[\nabla\times\left(g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\right)\right]\cdot\left[\nabla\phi + g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\right].\end{aligned} \tag{286}$$

Next, we work in cylindrical coordinates and obtain

$$g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2} = \frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_Z\hat{\mathbf{R}} - \frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_R\hat{\mathbf{Z}} \tag{287}$$

and

$$\nabla\times\left[g\frac{\nabla\phi\times\nabla\Psi}{|\nabla\Psi|^2}\right]=\left[\frac{\partial}{\partial Z}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_Z\right)+\frac{\partial}{\partial R}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_R\right)\right]\hat{\phi}. \tag{288}$$

Using Eq. (288), Eq. (286) is written as

$$S=\left[\frac{\partial}{\partial Z}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_Z\right)+\frac{\partial}{\partial R}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_R\right)\right]\frac{1}{R}. \tag{289}$$

The two partial derivatives appearing in the above equation can be calculated to give

$$\begin{aligned}\frac{\partial}{\partial Z}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_Z\right) &= \frac{1}{\Psi_R^2+\Psi_Z^2}\frac{1}{R}(g\Psi_Z)_Z-\frac{2\Psi_R\Psi_{RZ}+2\Psi_Z\Psi_{ZZ}}{[\Psi_R^2+\Psi_Z^2]^2}\frac{g}{R}\Psi_Z\\ &= \frac{1}{\Psi_R^2+\Psi_Z^2}\frac{g\Psi_{ZZ}+g'\Psi_Z^2}{R}-\frac{2\Psi_R\Psi_{RZ}+2\Psi_Z\Psi_{ZZ}}{[\Psi_R^2+\Psi_Z^2]^2}\frac{g}{R}\Psi_Z.\end{aligned} \tag{290}$$

$$\begin{aligned}\frac{\partial}{\partial R}\left(\frac{1}{|\nabla\Psi|^2}\frac{g}{R}\Psi_R\right) &= \frac{1}{\Psi_R^2+\Psi_Z^2}\left(\frac{g}{R}\Psi_R\right)_R-\frac{2\Psi_R\Psi_{RR}+2\Psi_Z\Psi_{ZR}}{[\Psi_R^2+\Psi_Z^2]^2}\frac{g}{R}\Psi_R\\ &= \frac{1}{\Psi_R^2+\Psi_Z^2}\left(\frac{g}{R}\Psi_{RR}+\Psi_R\frac{g'\Psi_R R-g}{R^2}\right)-\frac{2\Psi_R\Psi_{RR}+2\Psi_Z\Psi_{ZR}}{[\Psi_R^2+\Psi_Z^2]^2}\frac{g}{R}\Psi_R.\end{aligned} \tag{291}$$

Using these, we obtain

$$S=\frac{1}{\Psi_R^2+\Psi_Z^2}\frac{1}{R^2}\left(g\Psi_{ZZ}+g'\Psi_Z^2+g\Psi_{RR}+\Psi_R\frac{g'\Psi_R R-g}{R}\right)-\frac{1}{[\Psi_R^2+\Psi_Z^2]^2}\frac{g}{R^2}(4\Psi_R\Psi_{RZ}\Psi_Z+2\Psi_Z^2\Psi_{ZZ}+2\Psi_R^2\Psi_{RR}). \tag{292}$$

(The above results for S has been verified by using Mathematica Software.) The results calculated by using Eq. (292) are plotted in Fig. 32.

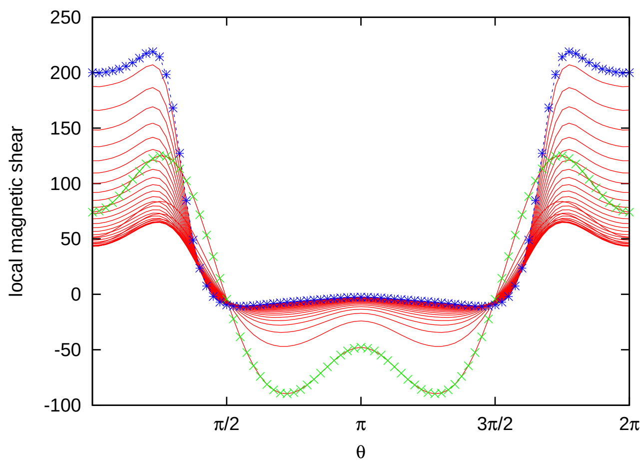

Figure 32. The local magnetic shear S as a function of the poloidal angle. The different lines corresponds to the shear on different magnetic surfaces. The stars correspond to the values of the shear on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is a Solovev equilibrium.

Next, we consider the calculation of S in the flux coordinates system $(\psi,\ \theta,\ \phi)$. The $\mathbf{B}_0\times\nabla\Psi$ term can be written as

$$\begin{aligned}\mathbf{B}_0\times\nabla\Psi &= [\nabla\Psi\times\nabla\phi+g\nabla\phi]\times\nabla\Psi\\ &= \Psi'^2|\nabla\psi|^2\nabla\phi+g\Psi'\nabla\phi\times\nabla\psi\\ &= \Psi'^2|\nabla\psi|^2\nabla\phi+g\Psi'\mathcal{J}^{-1}\left(-\frac{\mathcal{J}^2\nabla\psi\cdot\nabla\theta}{R^2}\nabla\psi+\frac{\mathcal{J}^2|\nabla\psi|^2}{R^2}\nabla\theta\right)\\ &= -\Psi'\frac{g\mathcal{J}\nabla\psi\cdot\nabla\theta}{R^2}\nabla\psi+\Psi'\frac{g\mathcal{J}|\nabla\psi|^2}{R^2}\nabla\theta+\Psi'^2|\nabla\psi|^2\nabla\phi\end{aligned} \tag{293}$$

$$\Rightarrow\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2}=-\frac{g\mathcal{J}}{\Psi'R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\nabla\psi+\frac{g\mathcal{J}}{\Psi'R^2}\nabla\theta+\nabla\phi. \tag{294}$$

By using the curl formula in generalized coordinates (ψ,θ,ϕ), we obtain

$$\begin{aligned}\nabla\times\frac{\mathbf{B}_0\times\nabla\Psi}{|\nabla\Psi|^2} &= \nabla\times\left(-\frac{g\mathcal{J}}{\Psi'R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\nabla\psi+\frac{g\mathcal{J}}{\Psi'R^2}\nabla\theta+\nabla\phi\right)\\ &= \left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi'R^2}\right)+\frac{\partial}{\partial\theta}\left(\frac{g\mathcal{J}}{\Psi'R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\right)\right]\nabla\psi\times\nabla\theta\end{aligned} \tag{295}$$

Using Eqs. (294) and (295), the negative local magnetic shear [Eq. (284)] is written as

$$\begin{aligned} S &= \left\{\left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi' R^2}\right)+\frac{\partial}{\partial\theta}\left(\frac{g\mathcal{J}}{\Psi' R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\right)\right]\nabla\psi\times\nabla\theta\right\}\cdot\left(-\frac{g\mathcal{J}}{\Psi' R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\nabla\psi+\frac{g\mathcal{J}}{\Psi' R^2}\nabla\theta+\nabla\phi\right) \\ &= \left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi' R^2}\right)+\frac{\partial}{\partial\theta}\left(\frac{g\mathcal{J}}{\Psi' R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\right)\right]\nabla\psi\times\nabla\theta\cdot\nabla\phi \\ &= \left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi' R^2}\right)+\frac{\partial}{\partial\theta}\left(\frac{g\mathcal{J}}{\Psi' R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\right)\right]\mathcal{J}^{-1}. \end{aligned} \tag{296}$$

Note that the partial derivatives $\partial/\partial\psi$ and $\partial/\partial\theta$ in Eq. (296) is taken in the (ψ,θ,ϕ) coordinates and they are usually different from their counterparts in (ψ,θ,ζ) coordinates. In Eq. (296), the partial derivatives are operating on equilibrium quantity, which is independent of ϕ and ζ. In this case, the partial derivatives in the two sets of coordinates are equal to each other. Figure 33 plots the poloidal dependence of local magnetic shear S and $|\nabla\Psi|^2 S$.

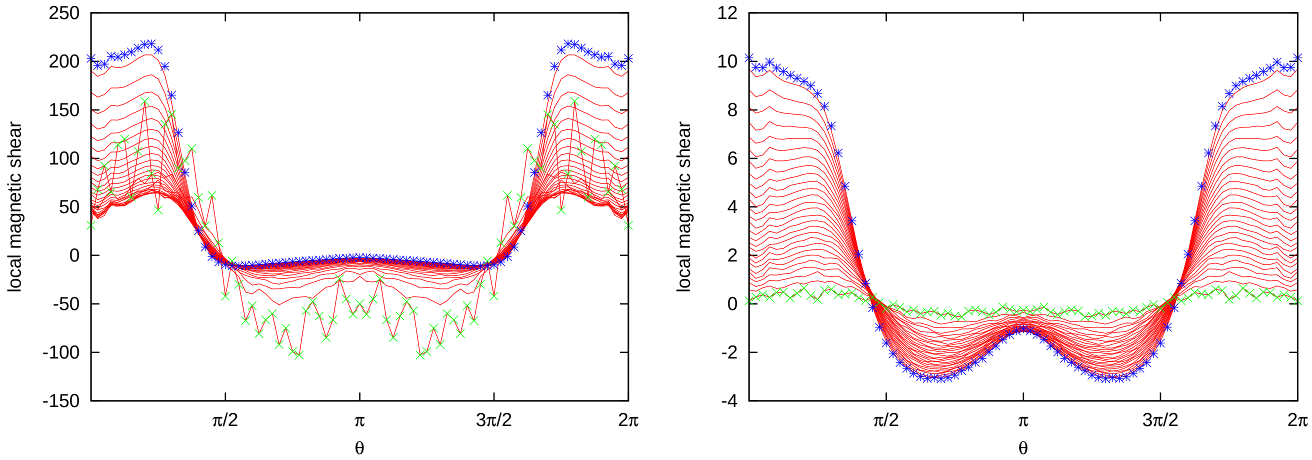

Figure 33. The local magnetic shear S (left) and $|\nabla\Psi|^2 S$ (right) as a function of the poloidal angle. The different lines corresponds to the shear on different magnetic surfaces. The stars correspond to the values of the shear on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is a Solovev equilibrium.

Next, let us examine the flux surface average of S, which is written as

$$\begin{aligned} \langle S\rangle &= \frac{\int_0^{2\pi} S\mathcal{J}d\theta}{\int_0^{2\pi}\mathcal{J}d\theta} \\ &= \frac{1}{\int_0^{2\pi}\mathcal{J}d\theta}\int_0^{2\pi}\left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi' R^2}\right)+\frac{\partial}{\partial\theta}\left(\frac{g\mathcal{J}}{\Psi' R^2}\frac{\nabla\psi\cdot\nabla\theta}{|\nabla\psi|^2}\right)\right]d\theta \\ &= \frac{1}{\int_0^{2\pi}\mathcal{J}d\theta}\int_0^{2\pi}\left[\frac{\partial}{\partial\psi}\left(\frac{g\mathcal{J}}{\Psi' R^2}\right)\right]d\theta \\ &= \frac{1}{\int_0^{2\pi}\mathcal{J}d\theta}\frac{\partial}{\partial\psi}\int_0^{2\pi}\frac{g\mathcal{J}}{\Psi' R^2}d\theta \end{aligned} \tag{297}$$

Note that the global safety factor is given by

$$q(\psi)=-\frac{1}{2\pi}\int_0^{2\pi}\frac{g\mathcal{J}}{\Psi' R^2}d\theta. \tag{298}$$

Using this, equation (297) is written as

$$\langle S\rangle=-\frac{2\pi}{\int_0^{2\pi}\mathcal{J}d\theta}\frac{dq(\psi)}{d\psi}, \tag{299}$$

Equation (299) provides a way to verify the correctness of the numerical implementation of S. Figure 34 compares $dq/d\psi$ with $-\langle S\rangle\left(\int_0^{2\pi}\mathcal{J}d\theta\right)/(2\pi)$, which shows that the two results agree with each other well.

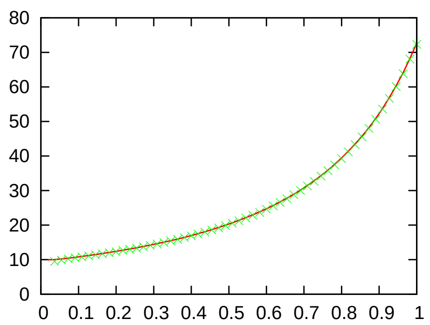

Figure 34. $dq/d\psi$ (solid line) and $-\langle S\rangle\int_0^{2\pi}\mathcal{J}d\theta/(2\pi)$ (plus mark) as a function of the radial coordinate $\overline{\Psi}$ ($\overline{\Psi}$ is the normalized poloidal magnetic flux). The equilibrium is a Solovev equilibrium.

9.6 Formula for matrix elements C_{22}

Next, consider the calculation of matrix elements C_{22}. In cylindrical coordinates, we have

$$\begin{aligned}\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) &= \nabla\cdot\left(\frac{\Psi_R}{|\nabla\Psi|^2}\mathbf{e}_R+\frac{\Psi_Z}{|\nabla\Psi|^2}\mathbf{e}_Z\right)\\ &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\Psi_R}{|\nabla\Psi|^2}\right)+\frac{\partial}{\partial Z}\left(\frac{\Psi_Z}{|\nabla\Psi|^2}\right)\\ &= \frac{\Psi_R}{R|\nabla\Psi|^2}+\frac{\partial}{\partial R}\left(\frac{\Psi_R}{|\nabla\Psi|^2}\right)+\Psi_{ZZ}\frac{1}{|\nabla\Psi|^2}+\Psi_Z\left[-\frac{1}{|\nabla\Psi|^4}(2\Psi_R\Psi_{RZ}+2\Psi_Z\Psi_{ZZ})\right]\\ &= \frac{\Psi_R}{R|\nabla\Psi|^2}+\Psi_{RR}\left(\frac{1}{|\nabla\Psi|^2}\right)+\Psi_R\left[-\frac{1}{|\nabla\Psi|^4}(2\Psi_R\Psi_{RR}+2\Psi_Z\Psi_{ZR})\right]+\Psi_{ZZ}\frac{1}{|\nabla\Psi|^2}+\Psi_Z\left[-\frac{1}{|\nabla\Psi|^4}(2\Psi_R\Psi_{RZ}+2\Psi_Z\Psi_{ZZ})\right]\\ &= \frac{\Psi_R}{R|\nabla\Psi|^2}+\frac{\Psi_{RR}+\Psi_{ZZ}}{|\nabla\Psi|^2}-\frac{1}{|\nabla\Psi|^4}(2\Psi_R\Psi_R\Psi_{RR}+4\Psi_R\Psi_Z\Psi_{ZR}+2\Psi_Z\Psi_Z\Psi_{ZZ}) \end{aligned} \tag{300}$$

Using this, C_{22} is written

$$\begin{aligned} C_{22} &= -|\nabla\Psi|^2\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)\\ &= -\frac{\Psi_R}{R}-(\Psi_{RR}+\Psi_{ZZ})+\frac{1}{|\nabla\Psi|^2}(2\Psi_R\Psi_R\Psi_{RR}+4\Psi_R\Psi_Z\Psi_{ZR}+2\Psi_Z\Psi_Z\Psi_{ZZ}). \end{aligned} \tag{301}$$

Equation (301) is used in GTAW to calculate C_{22}.

C_{22} can also be calculated in the magnetic surface coordinates.

$$\frac{C_{22}}{\Psi'|\nabla\psi|^2}=-\Psi'\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right), \tag{302}$$

the term on the r.h.s of the above equation is written as

$$\begin{aligned}\nabla\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\right) &= \nabla\cdot\left(\frac{\nabla\Psi}{R^2}\frac{R^2}{|\nabla\Psi|^2}\right)\\ &= \frac{R^2}{|\nabla\Psi|^2}\nabla\cdot\left(\frac{\nabla\Psi}{R^2}\right)+\frac{\nabla\Psi}{R^2}\cdot\nabla\left(\frac{R^2}{|\nabla\Psi|^2}\right). \end{aligned} \tag{303}$$

The first term on the r.h.s of Eq. (303) is written as

$$\begin{aligned}\frac{R^2}{|\nabla\Psi|^2}\nabla\cdot\left(\frac{\nabla\Psi}{R^2}\right) &= \frac{1}{|\nabla\Psi|^2}\triangle^*\Psi\\ &= \frac{1}{|\nabla\Psi|^2}\left[-\mu_0R^2\frac{dp_0}{d\Psi}-\frac{dg}{d\Psi}g(\Psi)\right]. \end{aligned} \tag{304}$$

The second term on the r.h.s of Eq. (303) is written as

$$\frac{\nabla\Psi}{R^2}\cdot\nabla\left(\frac{R^2}{|\nabla\Psi|^2}\right) = \frac{\Psi'}{R^2}\left[|\nabla\psi|^2\frac{\partial}{\partial\psi}\left(\frac{R^2}{|\nabla\Psi|^2}\right)+(\nabla\theta\cdot\nabla\psi)\frac{\partial}{\partial\theta}\left(\frac{R^2}{|\nabla\Psi|^2}\right)\right] \tag{305}$$

Using these, Eq. (302) is finally written as

$$\frac{C_{22}}{\Psi'|\nabla\psi|^2} = \Psi'\left[\mu_0\frac{dp_0}{d\Psi}\frac{R^2}{|\nabla\Psi|^2}+\frac{dg}{d\Psi}g(\Psi)\frac{1}{|\nabla\Psi|^2}\right]-\Psi'^2\left[\frac{|\nabla\psi|^2}{R^2}\frac{\partial}{\partial\psi}\left(\frac{R^2}{|\nabla\Psi|^2}\right)+\frac{\nabla\theta\cdot\nabla\psi}{R^2}\frac{\partial}{\partial\theta}\left(\frac{R^2}{|\nabla\Psi|^2}\right)\right] \tag{306}$$

10 misc contents

10.1 Relation of plasma displacement with experimental measurements

In experiments, the beam emission spectroscopy (BES) and microwave reflectometer can measure electron density fluctuation. The electron cyclotron emission (ECE) radiometer can measure electron temperature fluctuation. In the ideal MHD theory, we assume that $n_{e0}=n_{i0}$, $n_{e1}=n_{i1}$, and the mass density is given approximately by $\rho_m=n_im_i$. Then the linearized continuity equation, $\rho_1=-\boldsymbol{\xi}_1\cdot\nabla\rho_0-\rho_0\nabla\cdot\boldsymbol{\xi}_1$, is written as

$$n_{e1}=-\boldsymbol{\xi}\cdot\nabla n_{e0}-n_{e0}\nabla\cdot\boldsymbol{\xi}, \tag{307}$$

which gives relationship between the density fluctuation and plasma displacement. Similarly, in the ideal MHD theory, we assume that $T_{e0}=T_{i0}$, $T_{e1}=T_{i1}$, and $p=n_eT_e+n_iT_i=2n_eT_e$. Then the linearized equation of state, $p_1=-\boldsymbol{\xi}\cdot\nabla p_0-\gamma p_0\nabla\cdot\boldsymbol{\xi}$, is written as

$$n_0T_1+n_1T_0=-\boldsymbol{\xi}\cdot(T_0\nabla n_0+n_0\nabla T_0)-\gamma n_0T_0\nabla\cdot\boldsymbol{\xi}, \tag{308}$$

Using Eq. () to eliminate n_1, Eq. () is written as

$$\Longrightarrow n_0 T_1 - \boldsymbol{\xi} \cdot T_0 \nabla n_0 - n_0 T_0 \nabla \cdot \boldsymbol{\xi} = -\boldsymbol{\xi} \cdot T_0 \nabla n_0 - \boldsymbol{\xi} \cdot n_0 \nabla T_0 - \gamma n_0 T_0 \nabla \cdot \boldsymbol{\xi}, \tag{309}$$

$$\Longrightarrow T_1 = -\boldsymbol{\xi} \cdot \nabla T_0 - (\gamma - 1) T_0 \nabla \cdot \boldsymbol{\xi}, \tag{310}$$

$$\Longrightarrow \frac{T_1}{T_0} = -\boldsymbol{\xi} \cdot \frac{\nabla T_0}{T_0} - (\gamma - 1) \nabla \cdot \boldsymbol{\xi}, \tag{311}$$

The continuity equation is written as

$$\rho_1 = -\rho_0' \xi_\psi - p_0 \nabla \cdot \boldsymbol{\xi}, \tag{312}$$

Neglecting the compressible term, the above equation is written as

$$\rho_1 = -\rho_0' \xi_\psi. \tag{313}$$

Using $\rho \approx n_i m_i$ and $n_i = n_e$, the above equation is written as

$$n_{e1} = -n_{e0}' \xi_\psi. \tag{314}$$

Equation (314) relates the radial displacement obtained from a eigenvalue code with the density perturbation n_{e1}, which can be measured by the reflectometer in experiments.

We know that the radial plasma displacement ξ_ψ is related to the perturbed thermal pressure through the relation:

$$p_1 = -p_0' \xi_\psi - \gamma p_0 \nabla \cdot \boldsymbol{\xi}, \tag{315}$$

Neglecting the compressible term, the above equation is written as

$$p_1 = -p_0' \xi_\psi \tag{316}$$

10.2 Expression of $k_\parallel$ and k_θ

Next, we derive the expression for the parallel and poloidal wave-number of a perturbation of the form

$$\delta A = \delta A_0(\psi) \exp[i(m\theta - n\zeta - \omega t)], \tag{317}$$

where (ψ, θ, ζ) are the flux coordinators, with θ and ζ being the generalized poloidal and toroidal angles. The parallel (to the local equilibrium magnetic field) wave vector, $k_\parallel$, is defined by

$$k_\parallel = \frac{\Delta \mathrm{ph}}{\Delta l}, \tag{318}$$

where Δph is the phase angle change of the perturbation when moving a distance of Δl along the local equilibrium magnetic field. According to Eq. (317), the phase change can be written as

$$\Delta \mathrm{ph} = m \Delta\theta - n \Delta\zeta, \tag{319}$$

where $\Delta\zeta$ and $\Delta\theta$ are the change in the toroidal and poloidal angles when we move a distance of Δl along the magnetic field. Use Eq. (319) in Eq. (318), giving

$$k_\parallel = \frac{m\Delta\theta - n\Delta\zeta}{\Delta l}. \tag{320}$$

Noting that the safety factor is given by

$$q = \frac{\Delta\zeta}{\Delta\theta} \tag{321}$$

(which is exact since we are using flux coordinator, in which magnetic field lines are straight on (θ, ζ) plane), Eq. (320) is written as

$$k_\parallel = \frac{m/q - n}{\Delta l / \Delta\zeta}. \tag{322}$$

In the approximation of large aspect ratio, Δl can be approximated by $\Delta l \approx R_0 \Delta\zeta$, where R_0 is the major radius of the magnetic axis. Using this, the above equation is written as

$$k_\parallel \approx \frac{m - nq}{qR_0}. \tag{323}$$

(Remarks: I should use the exact expression for Δl to derive an exact expression for $k_\parallel$, I will do this later.) Equation (323) indicates that $k_\parallel$ is zero on the resonant surface.

Similarly, the component of the wave vector along the θ direction is written as

$$\begin{aligned} k_\theta &= \frac{\Delta \mathrm{ph}}{\Delta l_p} \\ &= \frac{m\Delta\theta - n \cdot 0}{\Delta l_p}, \end{aligned} \tag{324}$$

where Δl_p is the poloidal arc length when the poloidal angle changes by $\Delta\theta$. If the equal-arc poloidal angle is used, then $\Delta l_p = L_p/(2\pi)\Delta\theta$, where L_p is the poloidal circumference of the magnetic surface. Using this, Eq. (324) is written as

$$k_\theta = \frac{m}{L_p/2\pi}. \tag{325}$$

If a circular flux surface is assumed, then the above equation is written as

$$k_\theta = \frac{m}{r},$$

where r is the radius of the flux surface.

10.3 Two-dimensional mode structures on poloidal plane

Consider a harmonic with poloidal mode number m and toroidal mode number n,

$$\delta\phi(r,\theta,\varphi,t) = A(r)\sin[m\theta + n\varphi + \omega t + \alpha(r)]. \tag{326}$$

Choose a radial profile of the amplitude

$$A(r) = \exp\left(-\frac{(r-r_s)^2}{\Delta^2}\right). \tag{327}$$

Figure 35 plots the two-dimensional mode structures on the poloidal plane for two profiles of the radial phase variation given by

$$\alpha = 0, \tag{328}$$

and

$$\alpha = (r - r_s)\frac{2\pi}{8}, \tag{329}$$

respectively. Note that, compared with the case of $\alpha = 0$ (no radial phase variation), the radial phase variation given by Eq. (329) influence the mode structure on the poloidal plane, generating the so-called mode twist or shear[11], as shown by the left figure of Fig. 35.

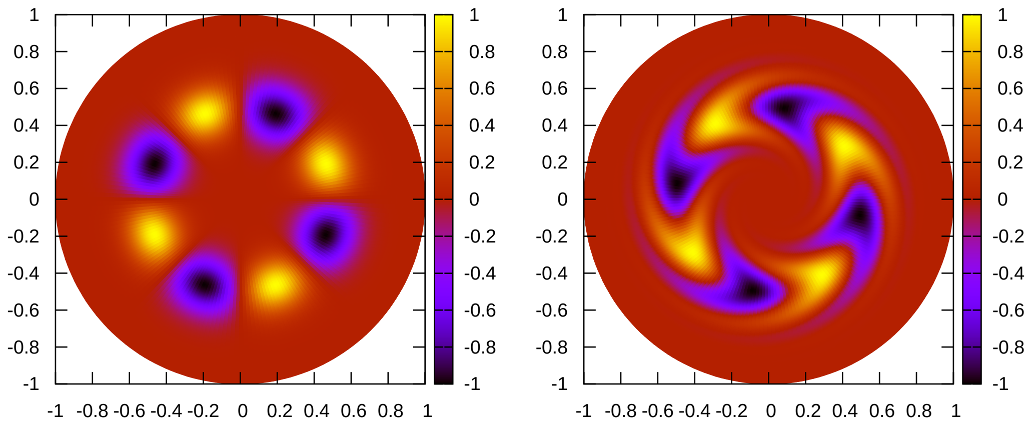

Figure 35. Two dimensional structure (on $\varphi = 0$ plane) given by Eqs. (326) and (327) with $m = 4$, $r_s = 0.5$, $\Delta = \sqrt{0.02}$, $t = 0$. Left figure is for $\alpha = 0$ and right figure is for α given by Eq. (334). The mode propagates (rotates) in the clockwise direction on the poloidal plane (the zero point of θ coordinate is at the low-field-side of the midplane and the positive direction is in the anti-clockwise direction). A GIF animation of the time evolution of the mode can be found at http://theory.ipp.ac.cn/~yj/figures/mode_rotation3.gif

Consider a mode composed of two poloidal harmonics

$$\delta\phi(r,\theta,\varphi,t) = A_1(r)\sin(m_1\theta + n\varphi + \omega t + \alpha) + A_2(r)\sin(m_2\theta + n\varphi + \omega t + \alpha), \tag{330}$$

where m_1 and m_2 are the poloidal mode number of the two poloidal harmonics. Consider the case $m_2 = m_1 + 1 = 5$. Then at the high field side of the midplane ($\theta = \pi$) of $\varphi = 0$ poloidal plane, equation (330) is written

$$\delta\phi(r,\theta,\varphi,t) = A_1\sin(\omega t + \alpha) + A_1\sin(\pi + \omega t + \alpha) = (A_1 - A_2)\sin(\omega t + \alpha). \tag{331}$$

At the low field side of the midplane ($\theta = 0$) of $\varphi = 0$ poloidal plane, equation (330) is written

$$\delta\phi(r,\theta,\varphi,t) = A_1\sin(\omega t + \alpha) + A_1\sin(\omega t + \alpha) = (A_1 + A_2)\sin(\omega t + \alpha). \tag{332}$$

Equations (331) and (332) indicates the amplitude of the mode at the low field side is larger than that at the strong field side, i.e., the mode exhibits a ballooning structure.

For a radial profile given by

$$A_1(r) = 2A_2(r) = \exp\left(-\frac{(r-r_s)^2}{\Delta^2}\right). \tag{333}$$

and an initial phase $\alpha = 0$, Figure 36 plots the two-dimensional structure of the mode on the poloidal plane. The inital phase α can have radial varation and this has effects on the 2D structure of the mode. For instance, α is chosen to be of the form

$$\alpha = \alpha(r) = (r - r_s)\frac{2\pi}{8}. \tag{334}$$

The resulting 2D mode structure is given in the right figure of Fig. 36, where the so-called mode shear can be seen[11].

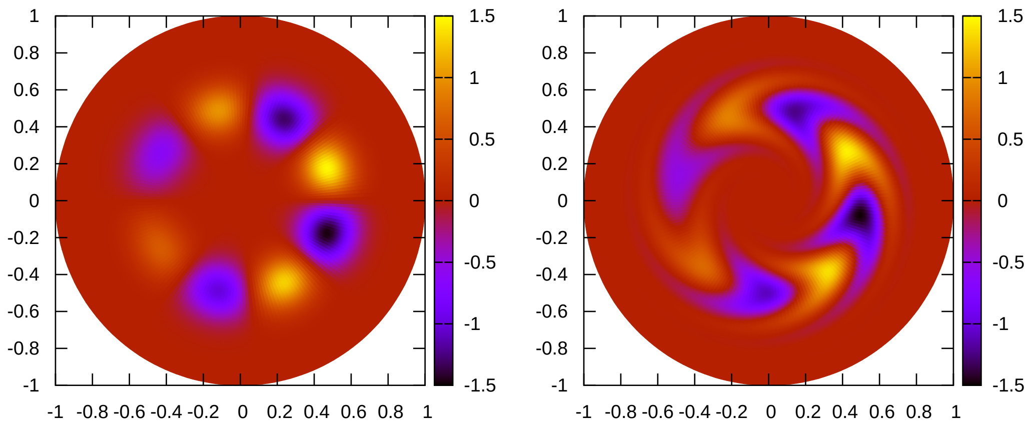

Figure 36. Two dimensional structure (on $\varphi = 0$ plane) given by Eqs. (330) and (333) with $m_1 = m_2 - 1 = 4$, $r_s = 0.5$, $\Delta = \sqrt{0.02}$, $t = 0$. Left figure is for $\alpha = 0$ and right figure is for α given by Eq. (334). Note that the mode amplitude at the low field side is larger than that at the high field side. The mode propagates (rotates) in the clockwise direction on the poloidal plane. A GIF animation of the time evolution of the mode can be found at http://theory.ipp.ac.cn/~yj/figures/ballooning_animation.gif

10.4 Magnetic islands

to be done.

10.5 proof: The derivation of Eq. (107)

Using Eq. (65), we obtain

$$\begin{aligned}
\nabla \times \mathbf{B}_1 &= \nabla \times \left(\frac{Q_\psi}{|\nabla\psi|^2}\nabla\psi + \frac{Q_s}{|\nabla\psi|^2}(\mathbf{B}_0 \times \nabla\psi) + \frac{Q_b}{B^2}\mathbf{B}_0 \right) \\
&= \nabla\frac{Q_\psi}{|\nabla\psi|^2} \times \nabla\psi + 0 + \nabla\frac{Q_s}{|\nabla\psi|^2} \times (\mathbf{B}_0 \times \nabla\psi) + \frac{Q_s}{|\nabla\psi|^2}\nabla \times (\mathbf{B}_0 \times \nabla\psi) + \nabla\frac{Q_b}{B_0^2} \times \mathbf{B}_0 + \frac{Q_b}{B_0^2}\mu_0\mathbf{J}_0 \\
&= \nabla\frac{Q_\psi}{|\nabla\psi|^2} \times \nabla\psi - \left(\mathbf{B}_0 \cdot \nabla\frac{Q_s}{|\nabla\psi|^2} \right)\nabla\psi + \left(\nabla\frac{Q_s}{|\nabla\psi|^2} \cdot \nabla\psi \right)\mathbf{B}_0 + \frac{Q_s}{|\nabla\psi|^2}\nabla \times (\mathbf{B}_0 \times \nabla\psi) + \nabla\frac{Q_b}{B_0^2} \times \mathbf{B}_0 + \\
&\quad \frac{Q_b}{B_0^2}\mu_0\mathbf{J}_0
\end{aligned} \tag{335}$$

Using this, the second term on the right-hand side of Eq. (106) is written

$$\begin{aligned}
\nabla\psi \cdot \mu_0^{-1}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 &= \mu_0^{-1}\mathbf{B}_0 \cdot \nabla\psi \times (\nabla \times \mathbf{B}_1) \\
&= \mu_0^{-1}\mathbf{B}_0 \cdot \nabla\psi \times \left(\nabla\frac{Q_\psi}{|\nabla\psi|^2} \times \nabla\psi + \frac{Q_s}{|\nabla\psi|^2}\nabla \times (\mathbf{B}_0 \times \nabla\psi) + \nabla\frac{Q_b}{B_0^2} \times \mathbf{B}_0 + \frac{Q_b}{B_0^2}\mu_0\mathbf{J}_0 \right).
\end{aligned} \tag{336}$$

The first term of Eq. (336) is written

$$\begin{aligned}
\mu_0^{-1}\mathbf{B}_0 \cdot \nabla\psi \times \left(\nabla\frac{Q_\psi}{|\nabla\psi|^2} \times \nabla\psi \right) &= \mu_0^{-1}\mathbf{B}_0 \cdot \left[|\nabla\psi|^2 \nabla\frac{Q_\psi}{|\nabla\psi|^2} - \left(\nabla\psi \cdot \nabla\frac{Q_\psi}{|\nabla\psi|^2} \right)\nabla\psi \right] \\
&= \mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0 \cdot \nabla\frac{Q_\psi}{|\nabla\psi|^2}
\end{aligned} \tag{337}$$

The last equality is due to $\mathbf{B}_0 \cdot \nabla\psi = 0$. The second term of Eq. (336) is written

$$\begin{aligned}
\mu_0^{-1}\mathbf{B}_0 \cdot \nabla\psi \times \frac{Q_s}{|\nabla\psi|^2}\nabla \times (\mathbf{B}_0 \times \nabla\psi) &= \mu_0^{-1}\frac{Q_s}{|\nabla\psi|^2}\mathbf{B}_0 \cdot [\nabla \times (\nabla\psi \times \mathbf{B}_0)] \times \nabla\psi \\
&= \mu_0^{-1}\frac{Q_s}{|\nabla\psi|^2}\mathbf{B}_0 \cdot [-\mathbf{B}_0(\nabla \cdot \nabla\psi) + (\mathbf{B}_0 \cdot \nabla)\nabla\psi - (\nabla\psi \cdot \nabla)\mathbf{B}_0] \times \nabla\psi \\
&= \mu_0^{-1}Q_s\frac{1}{|\nabla\psi|^2}\mathbf{B}_0 \cdot [(\mathbf{B}_0 \cdot \nabla)\nabla\psi - (\nabla\psi \cdot \nabla)\mathbf{B}_0] \times \nabla\psi \\
&= \mu_0^{-1}Q_s|\nabla\psi|^2 S
\end{aligned} \tag{338}$$

The last equality is due to that the coefficients before Q_s, i.e.,

$$\frac{1}{|\nabla\psi|^2}\mathbf{B}_0\cdot[(\mathbf{B}_0\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\mathbf{B}_0]\times\nabla\psi$$

is equal to $|\nabla\psi|^2S$ (refer to Sec. 10.7 for the proof). The third term of Eq. (336) is written

$$\begin{aligned}\mu_0^{-1}\mathbf{B}_0\cdot\nabla\psi\times\left(\nabla\frac{Q_b}{B_0^2}\times\mathbf{B}_0\right) &= \mu_0^{-1}\mathbf{B}_0\cdot\left[0-\left(\nabla\psi\cdot\nabla\frac{Q_b}{B_0^2}\right)\mathbf{B}_0\right]\\ &= -\mu_0^{-1}\left(\nabla\frac{Q_b}{B_0^2}\cdot\nabla\psi\right)B_0^2.\end{aligned}\tag{339}$$

Using Eqs. (337)-(339) in Eq. (336) yields

$$\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0=\mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0\cdot\nabla\frac{Q_\psi}{|\nabla\psi|^2}+\mu_0^{-1}Q_s|\nabla\psi|^2S-\mu_0^{-1}\left(\nabla\frac{Q_b}{B_0^2}\cdot\nabla\psi\right)B_0^2+\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot(\nabla\psi\times\mathbf{J}_0)\tag{340}$$

Now we calculate the $\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1$ term appearing in Eq. (106), which can be written as

$$\begin{aligned}\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1 &= -\mathbf{J}_0\cdot\nabla\psi\times\mathbf{B}_1\\ &= -\mathbf{J}_0\cdot\nabla\psi\times\left(\frac{Q_s}{|\nabla\psi|^2}(\mathbf{B}_0\times\nabla\psi)+\frac{Q_b}{B_0^2}\mathbf{B}_0\right)\\ &= -\mathbf{J}_0\cdot\left(\frac{Q_s}{|\nabla\psi|^2}\nabla\psi\times(\mathbf{B}_0\times\nabla\psi)+\frac{Q_b}{B_0^2}\nabla\psi\times\mathbf{B}_0\right)\\ &= -\mathbf{J}_0\cdot\left(\frac{Q_s}{|\nabla\psi|^2}|\nabla\psi|^2\mathbf{B}_0+\frac{Q_b}{B_0^2}\nabla\psi\times\mathbf{B}_0\right)\\ &= -\mathbf{J}_0\cdot\mathbf{B}_0Q_s-\mathbf{J}_0\cdot\left(\frac{Q_b}{B_0^2}\nabla\psi\times\mathbf{B}_0\right).\end{aligned}\tag{341}$$

Gathering the terms involving Q_b in Eqs. (340) and (341), we obtain

$$\begin{aligned}&-\mu_0^{-1}\left(\nabla\frac{Q_b}{B_0^2}\cdot\nabla\psi\right)B_0^2+\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot(\nabla\psi\times\mathbf{J}_0)+\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot(\nabla\psi\times\mathbf{J}_0)\\ &=-\left[\left(\frac{1}{B^2}\nabla Q_b+Q_b\nabla\frac{1}{B^2}\right)\cdot\nabla\psi\right]\mu_0^{-1}B_0^2+2\frac{Q_b}{B^2}\mathbf{B}_0\cdot[\nabla\psi\times\mathbf{J}_0]\\ &=-\mu_0^{-1}\nabla Q_b\cdot\nabla\psi-\mu_0^{-1}Q_bB_0^2\nabla\frac{1}{B_0^2}\cdot\nabla\psi+2\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot[\nabla\psi\times\mu_0^{-1}(\nabla\times\mathbf{B}_0)]\\ &=\mu_0^{-1}\left\{-\nabla Q_b\cdot\nabla\psi-Q_bB_0^2\nabla\frac{1}{B_0^2}\cdot\nabla\psi+2\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot[-(\mathbf{B}_0\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\mathbf{B}_0]\right\}\end{aligned}\tag{342}$$

The second term of Eq. (342) is written

$$\begin{aligned}-Q_bB_0^2(\nabla\psi\cdot\nabla)\frac{1}{B_0^2} &= -\left(0-Q_b\frac{1}{B_0^2}(\nabla\psi\cdot\nabla)B_0^2\right)\\ &= Q_b\frac{1}{B_0^2}(\nabla\psi\cdot\nabla)B_0^2\end{aligned}\tag{343}$$

The last term of Eq. (342) is written

$$\begin{aligned}-2\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot(\nabla\psi\cdot\nabla)\mathbf{B}_0 &= -\frac{Q_b}{B_0^2}(\nabla\psi\cdot\nabla)(\mathbf{B}_0\cdot\mathbf{B}_0)\\ &= -\frac{Q_b}{B_0^2}(\nabla\psi\cdot\nabla)B_0^2\end{aligned}\tag{344}$$

The terms in Eqs. (343) and (344) exactly cancel each other. Thus the expression in (342) now reduces to

$$\mu_0^{-1}\left\{-\nabla Q_b\cdot\nabla\psi+2\frac{Q_b}{B_0^2}\mathbf{B}_0\cdot[-(\mathbf{B}_0\cdot\nabla)\nabla\psi]\right\}\tag{345}$$

Noting that

$$\begin{aligned}\mathbf{B}_0\cdot[-(\mathbf{B}_0\cdot\nabla)\nabla\psi] &= -(\mathbf{B}_0\cdot\nabla)[\mathbf{B}_0\cdot\nabla\psi]+[(\mathbf{B}_0\cdot\nabla)\mathbf{B}_0]\cdot\nabla\psi\\ &= [(\mathbf{B}_0\cdot\nabla)\mathbf{B}_0]\cdot\nabla\psi,\end{aligned}\tag{346}$$

the expression (345) is further written as

$$\begin{aligned}&\mu_0^{-1}\left\{-\nabla Q_b\cdot\nabla\psi+2\frac{Q_b}{B^2}[\mathbf{B}_0\cdot\nabla\mathbf{B}_0]\cdot\nabla\psi\right\}\\ &=\mu_0^{-1}\left\{-\nabla Q_b\cdot\nabla\psi+2Q_b\frac{1}{B_0}\left[\frac{\mathbf{B}_0}{B_0}\cdot\nabla\left(\frac{\mathbf{B}_0}{B_0}B_0\right)\right]\cdot\nabla\psi\right\}\\ &=\mu_0^{-1}\left\{-\nabla Q_b\cdot\nabla\psi+2Q_b\frac{1}{B_0}[\mathbf{b}\cdot\nabla(\mathbf{b}B_0)]\cdot\nabla\psi\right\}\\ &=\mu_0^{-1}\{-\nabla Q_b\cdot\nabla\psi+2Q_b[\mathbf{b}\cdot\nabla(\mathbf{b})]\cdot\nabla\psi+0\}\\ &=\mu_0^{-1}\{-\nabla Q_b\cdot\nabla\psi+2Q_b\boldsymbol{\kappa}\cdot\nabla\psi\},\end{aligned}$$

where $\mathbf{b}=\mathbf{B}_0/B$ is the unit vector along the direction of equilibrium magnetic field, and $\boldsymbol{\kappa}=\mathbf{b}\cdot\nabla\mathbf{b}$ is the magnetic field curvature. Using these results, we obtain

$$\begin{aligned}&\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0+\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1\\&=\mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0\cdot\nabla\frac{Q_\psi}{|\nabla\psi|^2}+\mu_0^{-1}Q_s|\nabla\psi|^2S-\mathbf{J}_0\cdot\mathbf{B}_0Q_s+\mu_0^{-1}(-\nabla Q_b\cdot\nabla\psi+2Q_b\boldsymbol{\kappa}\cdot\nabla\psi).\end{aligned}\tag{347}$$

Using the above results, the radial component equation

$$-\omega^2\rho_{m0}\xi_\psi=-\nabla\psi\cdot\nabla p_1+\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0+\nabla\psi\cdot\mu_0^{-1}(\nabla\times\mathbf{B}_0)\times\mathbf{B}_1,\tag{348}$$

is written as

$$-\omega^2\rho_{m0}\xi_\psi=-\nabla\psi\cdot\nabla p_1-\mathbf{J}_0\cdot\mathbf{B}_0Q_s+\mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0\cdot\nabla\frac{Q_\psi}{|\nabla\psi|^2}+\mu_0^{-1}Q_s|\nabla\psi|^2S+\mu_0^{-1}(-\nabla Q_b\cdot\nabla\psi+2Q_b\boldsymbol{\kappa}\cdot\nabla\psi)\tag{349}$$

which can be arranged in the form

$$\begin{aligned}-\omega^2\rho_{m0}\xi_\psi&=-\nabla\psi\cdot\nabla(p_1+\mu_0^{-1}\mathbf{B}_1\cdot\mathbf{B}_0)+\mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{Q_\psi}{|\nabla\psi|^2}\right)\\&\quad+(\mu_0^{-1}|\nabla\psi|^2S-\mathbf{B}_0\cdot\mathbf{J}_0)Q_s+2\mu_0^{-1}\boldsymbol{\kappa}\cdot\nabla\psi Q_b.\end{aligned}\tag{350}$$

Define $P_1=p_1+\mu_0^{-1}\mathbf{B}_1\cdot\mathbf{B}_0$, the above equation is written as

$$\begin{aligned}-\omega^2\rho_{m0}\xi_\psi&=-\nabla\psi\cdot\nabla P_1+\mu_0^{-1}|\nabla\psi|^2\mathbf{B}_0\cdot\nabla\left(\frac{Q_\psi}{|\nabla\psi|^2}\right)\\&\quad+(\mu_0^{-1}|\nabla\psi|^2S-\mathbf{B}_0\cdot\mathbf{J}_0)Q_s+2\mu_0^{-1}\boldsymbol{\kappa}\cdot\nabla\psi Q_b,\end{aligned}\tag{351}$$

which is identical with Eq. (107).

10.6 proof

In this subsection, we prove that

$$\Pi\equiv(\mathbf{B}\times\nabla\psi)\cdot\left[Q_s\left(\mathbf{B}\cdot\nabla\frac{1}{|\nabla\psi|^2}\right)\mathbf{B}\times\nabla\psi+\frac{Q_s}{|\nabla\psi|^2}[-(\mathbf{B}\cdot\nabla\nabla\psi)\times\mathbf{B}+(\nabla\psi\cdot\nabla\mathbf{B})\times\mathbf{B}]\right],$$

is zero.

$$\begin{aligned}\Pi&=Q_s\left(\mathbf{B}\cdot\nabla\frac{1}{|\nabla\psi|^2}\right)B^2|\nabla\psi|^2+\frac{Q_s}{|\nabla\psi|^2}[-(\mathbf{B}\cdot\nabla\nabla\psi)\times\mathbf{B}\cdot(\mathbf{B}\times\nabla\psi)+(\nabla\psi\cdot\nabla\mathbf{B})\times\mathbf{B}\cdot(\mathbf{B}\times\nabla\psi)]\\&=Q_s\left(\mathbf{B}\cdot\nabla\frac{1}{|\nabla\psi|^2}\right)B^2|\nabla\psi|^2+\frac{Q_s}{|\nabla\psi|^2}[(\mathbf{B}\times\nabla\psi)\times\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\nabla\psi)-(\mathbf{B}\times\nabla\psi)\times\mathbf{B}\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=Q_s\left(\mathbf{B}\cdot\nabla\frac{1}{|\nabla\psi|^2}\right)B^2|\nabla\psi|^2+\frac{Q_s}{|\nabla\psi|^2}[(B^2\nabla\psi)\cdot(\mathbf{B}\cdot\nabla\nabla\psi)-B^2\nabla\psi\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=Q_s\left(\mathbf{B}\cdot\nabla\frac{1}{|\nabla\psi|^2}\right)B^2|\nabla\psi|^2+Q_s\frac{B^2}{|\nabla\psi|^2}[(\nabla\psi)\cdot(\mathbf{B}\cdot\nabla\nabla\psi)-(\nabla\psi)\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=Q_s\left(\mathbf{B}\cdot\nabla(|\nabla\psi|^2)\left(-\frac{1}{|\nabla\psi|^4}\right)\right)B^2|\nabla\psi|^2+Q_s\frac{B^2}{|\nabla\psi|^2}[(\nabla\psi)\cdot(\mathbf{B}\cdot\nabla\nabla\psi)-(\nabla\psi)\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=2Q_s\nabla\psi\cdot\mathbf{B}\cdot\nabla\nabla\psi\left(-\frac{B^2}{|\nabla\psi|^2}\right)+Q_s\frac{B^2}{|\nabla\psi|^2}[(\nabla\psi)\cdot(\mathbf{B}\cdot\nabla\nabla\psi)-(\nabla\psi)\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=-Q_s\frac{B^2}{|\nabla\psi|^2}\nabla\psi\cdot\mathbf{B}\cdot\nabla\nabla\psi-Q_s\frac{B^2}{|\nabla\psi|^2}(\nabla\psi)\cdot(\nabla\psi\cdot\nabla\mathbf{B})]\\&=-Q_s\frac{B^2}{|\nabla\psi|^2}\nabla\psi\cdot[\mathbf{B}\cdot\nabla\nabla\psi+(\nabla\psi\cdot\nabla\mathbf{B})]\\&=-Q_s\frac{B^2}{|\nabla\psi|^2}\nabla\psi\cdot[\mathbf{B}\cdot\nabla\nabla\psi+(\nabla\psi\cdot\nabla\mathbf{B})+\nabla\psi\times\nabla\times\mathbf{B}+\mathbf{B}\times\nabla\times\nabla\psi]\\&=-Q_s\frac{B^2}{|\nabla\psi|^2}\nabla\psi\cdot\nabla(\mathbf{B}\cdot\nabla\psi)\\&=0\end{aligned}\tag{352}$$

where the last second equality is due to the fact that

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{B}\cdot\nabla\mathbf{A}+\mathbf{A}\cdot\nabla\mathbf{B}+\mathbf{A}\times\nabla\times\mathbf{B}+\mathbf{B}\times\nabla\times\mathbf{A}\tag{353}$$

10.7 proof 2

We want to prove that

$$\mathbf{B}_0\cdot[(\mathbf{B}_0\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\mathbf{B}_0]\times\frac{\nabla\psi}{|\nabla\psi|^2}=|\nabla\psi|^2S,\tag{354}$$

where $S=\left(\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right)\cdot\nabla\times\left(\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right)$. The right-hand side of Eq. (354) is written as

$$\begin{aligned}
|\nabla\psi|^2\left(\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right)\cdot\nabla\times\left(\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right) &= (\mathbf{B}_0\times\nabla\psi)\cdot\left[-(\mathbf{B}_0\cdot\nabla)\frac{\nabla\psi}{|\nabla\psi|^2}+\left(\frac{\nabla\psi}{|\nabla\psi|^2}\cdot\nabla\right)\mathbf{B}_0\right]\\
&= -(\mathbf{B}_0\times\nabla\psi)\cdot(\mathbf{B}_0\cdot\nabla)\frac{\nabla\psi}{|\nabla\psi|^2}+\left(\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right)\cdot(\nabla\psi\cdot\nabla)\mathbf{B}_0\\
&= \left[(\mathbf{B}_0\cdot\nabla)\frac{\nabla\psi}{|\nabla\psi|^2}\times\nabla\psi\right]\cdot\mathbf{B}_0-\left[(\nabla\psi\cdot\nabla)\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right]\cdot\mathbf{B}_0 \qquad (355)\\
&= \left[\frac{1}{|\nabla\psi|^2}(\mathbf{B}_0\cdot\nabla)\nabla\psi\times\nabla\psi\right]\cdot\mathbf{B}_0-\left[(\nabla\psi\cdot\nabla)\mathbf{B}_0\times\frac{\nabla\psi}{|\nabla\psi|^2}\right]\cdot\mathbf{B}_0\\
&= \mathbf{B}_0\cdot[(\mathbf{B}_0\cdot\nabla)\nabla\psi-(\nabla\psi\cdot\nabla)\mathbf{B}_0]\times\frac{\nabla\psi}{|\nabla\psi|^2}, \qquad (356)
\end{aligned}$$

which is exactly the left-hand side of Eq. (354). Thus Eq. (354) is proved.

10.8 proof

Try to prove that

$$\nabla\cdot\frac{(\mathbf{B}_0\times\nabla\psi)}{B_0^2} = -2\boldsymbol{\kappa}\cdot\left(\frac{\mathbf{B}_0\times\nabla\psi}{B_0^2}\right). \qquad (357)$$

Proof:

$$\begin{aligned}
\nabla\cdot\frac{(\mathbf{B}_0\times\nabla\psi)}{B_0^2} &= \frac{1}{B_0^2}\nabla\cdot(\mathbf{B}_0\times\nabla\psi)+(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2}\\
&= \frac{1}{B_0^2}\nabla\psi\cdot\nabla\times\mathbf{B}_0+(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2}\\
&= \frac{1}{B_0^2}\nabla\psi\cdot\mu_0\mathbf{J}_0+(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2}\\
&= 0+(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2}. \qquad (358)
\end{aligned}$$

Using

$$(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2}=-\frac{2}{B_0^2}(\mathbf{B}_0\times\nabla\psi)\cdot\boldsymbol{\kappa} \qquad (359)$$

(proof of this is given in Sec. 10.9), Eq. (358) is written as

$$\nabla\cdot\frac{(\mathbf{B}_0\times\nabla\psi)}{B_0^2}=-\frac{2}{B_0^2}(\mathbf{B}_0\times\nabla\psi)\cdot\boldsymbol{\kappa} \qquad (360)$$

10.9 proof of $(\boldsymbol{B}\times\nabla\psi)\cdot\nabla\left(\frac{1}{\boldsymbol{B}^2}\right)=-2\boldsymbol{\kappa}\cdot(\boldsymbol{B}\times\nabla\psi)/\boldsymbol{B}^2$

The left-hand side of the equation is written as

$$\begin{aligned}
(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2} &= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot\nabla B_0\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot\nabla(\mathbf{B}_0\cdot\mathbf{b}) \qquad (361)
\end{aligned}$$

Using $\boldsymbol{\kappa}=\mathbf{b}\cdot\nabla\mathbf{b}=-\mathbf{b}\times\nabla\times\mathbf{b}$, the above equation is reduced to

$$\begin{aligned}
(\mathbf{B}_0\times\nabla\psi)\cdot\nabla\frac{1}{B_0^2} &= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[\mathbf{B}_0\times\nabla\times\mathbf{b}+\mathbf{b}\times\nabla\times\mathbf{B}_0+\mathbf{b}\cdot\nabla\mathbf{B}_0+\mathbf{B}_0\cdot\nabla\mathbf{b}]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[\mathbf{b}\times\nabla\times\mathbf{B}_0+\mathbf{b}\cdot\nabla\mathbf{B}_0]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[\mathbf{b}\times\mu_0\mathbf{J}_0+\mathbf{b}\cdot\nabla\mathbf{B}_0]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[\mathbf{b}\cdot\nabla\mathbf{B}_0]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[\mathbf{b}\cdot\nabla B_0\mathbf{b}]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[(\mathbf{b}\cdot\nabla B_0)\mathbf{b}+B_0\mathbf{b}\cdot\nabla\mathbf{b}]\\
&= -\frac{2}{B_0^3}(\mathbf{B}_0\times\nabla\psi)\cdot[B_0\mathbf{b}\cdot\nabla\mathbf{b}]\\
&= -\frac{2}{B_0^2}(\mathbf{B}_0\times\nabla\psi)\cdot\boldsymbol{\kappa}
\end{aligned}$$

Another way to prove that $(\mathbf{B}\times\nabla\psi)\cdot B^2\nabla\frac{1}{B^2}$ is equal to $-2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\psi)$: The difference of these two terms is

$$
\begin{aligned}
&-(\mathbf{B}\times\nabla\psi)\cdot B^2\nabla\frac{1}{B^2}-2\mathbf{K}\cdot(\mathbf{B}\times\nabla\psi)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(B^2\nabla\frac{1}{B^2}+2\mathbf{K}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(B^2\nabla\frac{1}{B^2}+2\frac{\mathbf{B}}{B}\cdot\nabla\frac{\mathbf{B}}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(B^2\nabla\frac{1}{B^2}+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\frac{1}{B}\mathbf{B}+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\frac{1}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(B^2\nabla\frac{1}{B^2}+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\frac{1}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(B^2\nabla B^2\left(-\frac{1}{B^4}\right)+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\frac{1}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(-\frac{1}{B^2}\nabla B^2+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\frac{1}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left(-\frac{1}{B^2}\nabla\mathbf{B}\cdot\mathbf{B}+\left(2\frac{\mathbf{B}}{B}\cdot\nabla\right)\mathbf{B}\frac{1}{B}\right)\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left[-\frac{1}{B^2}(2\mathbf{B}\times\nabla\times\mathbf{B}+2\mathbf{B}\cdot\nabla\mathbf{B})+\left(2\frac{\mathbf{B}}{B^2}\cdot\nabla\right)\mathbf{B}\right]\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left[-\frac{1}{B^2}2\mathbf{B}\times\nabla\times\mathbf{B}\right]\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left[-\frac{1}{B^2}2\mathbf{B}\times\mathbf{J}\right]\\
&=-(\mathbf{B}\times\nabla\psi)\cdot\left[-\frac{1}{B^2}2\nabla P\right]\\
&=(\mathbf{B}\times\nabla\psi)\cdot\left[-\frac{1}{B^2}2P'\nabla\psi\right]\\
&=0
\end{aligned}
$$

Thus we prove that

$$-Q_b(\mathbf{B}\times\nabla\psi)\cdot B^2\nabla\frac{1}{B^2}=2\boldsymbol{\kappa}\cdot(\mathbf{B}\times\nabla\psi)Q_b \tag{362}$$

10.10 proof

Try to prove that

$$\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)=-2\boldsymbol{\kappa}\cdot\nabla\Psi\frac{B^2}{|\nabla\Psi|^2}+\mu_0\frac{dp_0}{d\Psi}. \tag{363}$$

Proof: We have

$$\nabla p_0=\frac{dp_0}{d\Psi}\nabla\Psi \tag{364}$$

Thus

$$
\begin{aligned}
\mu_0\frac{dp_0}{d\Psi} &= \mu_0\frac{\nabla\Psi\cdot\nabla p_0}{|\nabla\Psi|^2}\\
&= \mu_0\frac{\nabla\Psi\cdot(\mathbf{J}_0\times\mathbf{B}_0)}{|\nabla\Psi|^2}\\
&= \frac{\nabla\Psi\cdot((\nabla\times\mathbf{B}_0)\times\mathbf{B}_0))}{|\nabla\Psi|^2}\\
&= -(\nabla\times\mathbf{B}_0)\cdot\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\times\mathbf{B}_0\right)\\
&= \nabla\cdot\left[\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\times\mathbf{B}_0\right)\times\mathbf{B}_0\right]-\mathbf{B}_0\cdot\nabla\times\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\times\mathbf{B}_0\right)\\
&= -\nabla\cdot\left(B_0^2\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)-\mathbf{B}_0\cdot\nabla\times\left(\frac{\nabla\Psi}{|\nabla\Psi|^2}\times\mathbf{B}_0\right)\\
&= -\nabla\cdot\left(B_0^2\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)-\mathbf{B}_0\cdot\left(-\mathbf{B}_0\nabla\cdot\frac{\nabla\Psi}{|\nabla\Psi|^2}+\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)\\
&= -\nabla\cdot\left(B_0^2\frac{\nabla\Psi}{|\nabla\Psi|^2}\right)+B_0^2\nabla\cdot\frac{\nabla\Psi}{|\nabla\Psi|^2}-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)\\
&= -\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla B_0^2-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)\\
&= -\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\cdot\mathbf{B}_0-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)\\
&= -\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot(2\mathbf{B}_0\times\nabla\times\mathbf{B}_0+2\mathbf{B}_0\cdot\nabla\mathbf{B}_0)-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)\\
&= -\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot(2\mathbf{B}_0\times\nabla\times\mathbf{B}_0)-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)-\frac{2\nabla\Psi}{|\nabla\Psi|^2}\cdot(\mathbf{B}_0\cdot\nabla B_0\mathbf{b})
\end{aligned} \tag{365}
$$

The last term on the right-hand side of Eq. (365) is written as

$$-\frac{2\nabla\Psi}{|\nabla\Psi|^2}\cdot(\mathbf{B}_0\cdot\nabla B_0\mathbf{b}) = -\frac{2\nabla\Psi}{|\nabla\Psi|^2}\cdot(B_0\mathbf{B}_0\cdot\nabla\mathbf{b}+(\mathbf{B}_0\cdot\nabla B)\mathbf{b})$$
$$= -\frac{2\nabla\Psi}{|\nabla\Psi|^2}\cdot(B_0\mathbf{B}_0\cdot\nabla\mathbf{b})$$
$$= -\frac{2B^2\nabla\Psi}{|\nabla\Psi|^2}\cdot\boldsymbol{\kappa}.$$

Using this, Eq. (365) is written as

$$\mu_0\frac{dp_0}{d\Psi} = \frac{2\nabla\Psi}{|\nabla\Psi|^2}\cdot((\nabla\times\mathbf{B}_0)\times\mathbf{B}_0)-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)-\frac{2B^2\nabla\Psi}{|\nabla\Psi|^2}\cdot\boldsymbol{\kappa}$$
$$= 2\mu_0\frac{dp_0}{d\Psi}-\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right)-\frac{2B^2\nabla\Psi}{|\nabla\Psi|^2}\cdot\boldsymbol{\kappa}, \quad (366)$$

i.e.,

$$\mathbf{B}_0\cdot\left(\mathbf{B}_0\cdot\nabla\frac{\nabla\Psi}{|\nabla\Psi|^2}-\frac{\nabla\Psi}{|\nabla\Psi|^2}\cdot\nabla\mathbf{B}_0\right) = \mu_0\frac{dp_0}{d\Psi}-\frac{2B^2\nabla\Psi}{|\nabla\Psi|^2}\cdot\boldsymbol{\kappa}, \quad (367)$$

thus Eq. (363) is proved.

10.11 Incompressible condition

The continuity equation can be written

$$\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\mathbf{u}+\mathbf{u}\cdot\nabla\rho=0, \quad (368)$$

which can be further written

$$\Rightarrow\frac{d\rho}{dt}=-\rho\nabla\cdot\mathbf{u}. \quad (369)$$

Then the incompressible condition $d\rho/dt=0$ reduces to that

$$\nabla\cdot\mathbf{u}=0 \quad (370)$$

On the other hand, equation (15) indicates that

$$\frac{d\rho}{dt}=\frac{1}{\gamma}\frac{\rho}{p}\frac{dp}{dt}, \quad (371)$$

which indicates that for incompressible plasma $dp/dt=0$.

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